Math 7240, Linear Algebraic Groups over Fields, Fall 2025
Homework

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from lecture 1

1. Let X be a k-space (that is, a functor from commutative k-algebras to sets). A field point of X is an element of $X(L)$ where L is a field extension of k. If $x \in X(L)$ and $y \in X(E)$ are field valued points of X, we say that $x \sim y$ if there exists a field extension M/k and morphisms of field extensions $\phi : E \to M$, $\psi : L \to M$ such that $X(\psi)(x) = X(\phi)(x)$ (in class we wrote this as $x_M = y_M$).

   Show that if $X = \operatorname{Spec} A$, then the equivalence classes of field valued points of X are in bijection with the prime ideals of A.

2. Recall that if X is a k-space which is of the form $X(R) = \operatorname{Hom}_{k-alg}(A, R)$ for some k-algebra A (i.e. X is a representable functor), we say that A is the coordinate ring of X and write $A = k[X]$. We also say in this case that X is the spectrum of A and write $X = \operatorname{Spec} A$. If X has this form, we say that X is an affine k-scheme.

   Show that Spec and $k[\ \_\ ]$ defines an equivalence of categories between the category of affine k-schemes (that is, the full subcategory of the category of k-spaces consisting of affine k-schemes) and the opposite of the category of k-algebras.

   note: you should prove this "from scratch," and not simply quote theorems from category theory!

3. (possibly challenging)

   (a) Show that if A, B are k-algebras, then $A \times B$ is the categorical product of A and B in the category of rings and $\operatorname{Spec}(A \times B)$ is the categorical coproduct of $\operatorname{Spec} A$ and $\operatorname{Spec} B$ in the category of k-spaces.

   (b) On the other hand, show that if $A_\lambda, \lambda \in \Lambda$ is a collection of nonzero k-algebras, then $\times_{\lambda\in\Lambda} A_\lambda$ is the categorical product of the A_λ's in the category of rings, but if Λ is infinite, $\operatorname{Spec}(\times_{\lambda\in\Lambda} A_\lambda$ is <u>not</u> the categorical coproduct of the spaces $\operatorname{Spec} A_\lambda$ in the category of k-spaces.

from lecture 2

4. Suppose $X = Z(f_1, \ldots, f_s)$ for $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ and $Y = Z(g_1, \ldots, g_t) \in k[y_1, \ldots, y_m]$ are finite type affine k-schemes. We think of these as sitting inside the affine spaces $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ where $\mathbb{A}^n$ has coordinate functions given by the x_i's and $\mathbb{A}^m$ has coordinate functions given by the y_i's.

   A morphism of affine schemes from X to Y is a collection of polynomials $\phi = (\phi_1, \ldots, \phi_m)$ with $\phi_i \in k[x_1, \ldots, x_n]$ (which we can think of as polynomial functions from $\mathbb{A}^n$ to $\mathbb{A}^m$), such that whenever we have an R-point of X, that is, $a = (a_1, \ldots, a_n) \in R^n$ such that $f_i(a) = 0$ for all i, we have $\phi(a) = (\phi_1(a), \phi_2(a), \ldots, \phi_m(a)) \in R^m$ is an R-point of Y.

   Show that morphisms of affine schemes $X \to Y$ are in bijection with natural transformations of functors from X to Y (considered as k-spaces).

5. Show that if the coordinate rings of X and Y are domains in the prior problem, if $\phi = (\phi_1, \ldots, \phi_m)$ with $\phi_i \in k[x_1, \ldots, x_n]$, then ϕ is a morphism from X to Y as affine schemes if and only if for every field extension L/k, we have an L-point of X, $a = (a_1, \ldots, a_n) \in L^n$ then $\phi(a) = (\phi_1(a), \phi_2(a), \ldots, \phi_m(a)) \in L^m$ is an L-point of Y.

6. (challenging) In the previous problem, show that instead of considering all field extensions L/k, it suffices to consider any single field extension L/k with L algebraically closed!