

## Structure of unipotent reps

Thm: If  $G$  is unipotent over a perfect field  $k$   $G \hookrightarrow GL_n$   
then we can find a basis where elements of  $G$  are strictly upper  $\Delta$ .

Lemma: If  $\rho: G \rightarrow GL(V)$  is any rep. of unipotent / perfect  $k$   
then  $\exists v \in V \setminus \{0\}$  which is  $G$ -fixed (i.e.  $gv = v$  for all  $g \in G(\bar{k})$ )

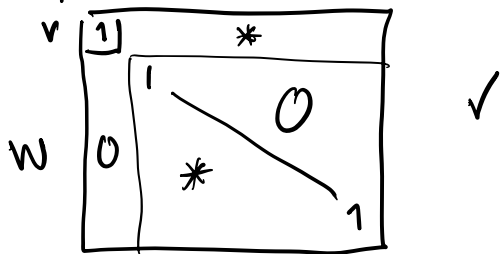
Lemma  $\Rightarrow$  Thm?

By induction on  $\dim V$       $\dim 1 \Rightarrow \text{rep} = \text{trivial}$

Induction step if  $v$  fixed by  $G$  then  $V / \langle v \rangle = W$

then  $W$  also a rep (quotients of reps are reps)

By induction, rep on  $W$  is lower  $\Delta$  w/ 1s on diag.



Pf. of Lemma: To find a fixed vector in  $V$  can assume  $V$  is red.

In fact: in this case we'll show  $V$  is a trivial rep.

Choose  $g \in G(\bar{k})$  we'll show  $\rho(g) = \text{id}_V$

By "linear algebra" can assume  $k = \bar{k}$  — point is to show that the

exercise w/ exercise 1 is  $\neq 0$ .

want to show  $\rho(g)=1$  will do this by writing  $g = g_s + g_n$  in  $\text{End}(V)$   
 $= 1 + g_n$

want:  $g_n = 0$

$$g_n = 0 \Leftrightarrow \text{tr}(\rho(g_n)T) = 0 \text{ all } T \in \text{End}(V)$$

But: if  $R = \text{subalg gen by } \rho(G(k)) \text{ in } \text{End}(V)$

then Jacob density  $\Rightarrow R = \text{End}(V)$   
(irreducibility)

$k[G(k)] \hookrightarrow \text{End}(V)$  so can consider  $T$  of form

$$\left\{ \sum_{g \in G(k)} x_g g \mid x_g \in k \right\} \quad \sum x_g \rho(g)$$

enough to show  $\text{tr}(\rho(g_n)\rho(h)) = 0$  all  $h \in G(k)$

$$\text{tr}((\rho(g)-1)\rho(h)) = \text{tr}(\rho(g)\rho(h) - \rho(h))$$

$$= \text{tr}(\rho(g)\rho(h)) - \text{tr}(\rho(h))$$

$$= \text{tr}(\rho(gu)) - \text{tr}(\rho(h))$$

$$\text{tr} \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} - \text{tr} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

$$= \dim V - \dim V = 0.$$

$$\Rightarrow \rho(g_n) = 0 \text{ all } g \Rightarrow \rho(g) = 1 + \rho(g_n) = 1 \text{ D.}$$

Soln:

Commutative structure theorem:

$G/k$  perfect  $\Rightarrow$

$$G \cong G_s \times G_u$$

comm.  $\uparrow$  all elements commute (sep)  $\uparrow$  unipotent.

Unipotent/perfect = simult. unipotent matrices.

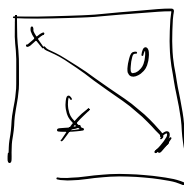
Remark: If  $G/k = \bar{k}$  comm,  $G_s \hookrightarrow GL(V)$  is diagonal after some change of basis of  $V$ .

Since each  $g \in G_s(k)$  separable as a lin. trans  $\Rightarrow$  diagonalizable but also  $G$  comm  $\Rightarrow$  simultaneously diagonalizable  $\square$ .

Fact (will come back to) in this case  $G_s^0$  connected comp. of 1 we have  $G_s^0 \cong G_m^n$  some  $n$  ( $k = \bar{k}$ )

Next target: Solvable groups

Method: define derived subgrp (commutator subgrp)



$$D(G) = [G, G] \quad G \supseteq D(G) \supseteq D(D(G)) \supseteq \dots \supseteq \dots \stackrel{?}{e} \text{ solvable.}$$

(Corrad 16.2)

Prop: let  $G$  be a smooth  $k$ -gp and  $X_1, \dots, X_n \neq \emptyset$  a collection of varieties (geom. integral finite type  $k$ -schemes)

and  $f_i: X_i \rightarrow G$  morphisms.  $e \in f_i(X_i)$   
 $\downarrow$   $\downarrow$  connected

then  $\exists!$  <sup>closed subgroup</sup> smooth  $k$ -subgroup  $H \leq G$  s.t.

$$H(\bar{k}) = \langle X_1(\bar{k}), \dots, X_n(\bar{k}) \rangle$$

and  $\exists$  finite list of pairs  $(i_1, m_1), (i_2, m_2), \dots, (i_r, m_r)$

$$\text{s.t. } X_{i_1} \times \dots \times X_{i_r} \rightarrow H \text{ via}$$

$$(X_1, \dots, X_n) \mapsto X_1^{m_1} X_2^{m_2} \dots X_r^{m_r} \text{ is surjective.}$$

Proof strategy

keep adding  $X_i$ 's to make dimension of products get bigger

$$(\dim X_1(\bar{k}))(\dim X_2(\bar{k}))(\dim X_3(\bar{k}))^{-1} \dots$$

keep going to grow dimension of prod.

$U$  product so far, can replace  $U$  by  ~~$UU^{-1}$  to assure  $e \in U$~~

one growth step, let  $H = \text{closure of product so far.}$   
 $U$

$$\cancel{U \subseteq UU}$$

$$U \subseteq X_i U$$

$$U \subseteq UU^{-1} \text{ define so } UU \subseteq \bar{U} = H$$

$$UU^{-1} \subseteq \bar{U} = H$$

$$\begin{array}{ccc} U \times U & \rightarrow & H \\ \cap & \searrow & \uparrow \\ H \times H & \rightarrow & G \end{array}$$

but  $U$  itself didn't need closure!

$$UU^{-1} = H$$

lemma: if  $U \subseteq H$  <sup>op variety</sup> dense in a LAG then  
 $H = UU^{-1}$  (on  $\text{rank} = 1$   $\bar{k}$ )

$$\underline{\text{Pf:}} \quad \text{wss: } h \in U U^{-1} \Leftrightarrow h U \cap U \neq \emptyset$$

$$\forall h \in H / \quad h = u v^{-1} \quad h v = u$$

but  $hU$  &  $U$  are both dense open  
 $\Rightarrow hU \cap U \neq \emptyset \quad \square$ .

$$\mathbb{D}G = \left\langle \text{im: } G \times G \rightarrow G \right\rangle$$

$$g, h \mapsto g h g^{-1} h^{-1}$$