

Fix a ring R (associative, unital, not necessarily commutative)

Let ${}_R\text{Mod}$ cat. of left R -modules
 Mod_R " right "

Remark ${}_R\text{Mod}$ is equivalent to the category Mod_{R^op}

unfair question: is ${}_R\text{Mod}$ isomorphic to Mod_{R^op} ?

Typical convention: Use Mod_R

$f: M \rightarrow N$ is a right R -module map
if $f(mr) = (fm)r$

Def We say that a sequence of maps

$A \xrightarrow{f} B \xrightarrow{g} C$ is exact (at B) if
im $f = \ker g$.

Def exact sequence of multiple maps

$A_{i+1} \rightarrow A_i \rightarrow A_{i-1} \dots$ means exact
at each A_i .

Def SES_R is the category whose objects
are short exact sequences and whose
morphisms are comm. diagrams

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A' & \rightarrow & B' & \rightarrow & C' \rightarrow 0 \end{array}$$

Things in common w/ Mod_R

Def Ab-Category is a category A together
with the structure of an Abelian group on
every hom set. $\text{Hom}_A(A, B)$ is an Ab gp

s.t. $f \cdot (g + g') = fg + fg'$
 $(g + g') h = gh + g'h.$

Observation SES_R is an Ab-Cat, (Mod_R is also)
(induced by add. maps in each component)

Def If A, B are Ab-Cats, a functor $F: A \rightarrow B$ is
additive if $\forall A, B \in \text{ob}(A)$, the map

$\text{Hom}_{\mathcal{A}}(A, B) \rightarrow \text{Hom}_{\mathcal{B}}(FA, FB)$ is an Ab-gp
 $(f: A \rightarrow B) \longmapsto (Ff: FA \rightarrow FB)$ homomorphism.

Exercise Suppose \mathcal{A} is an Ab-Cat, \mathcal{C} is a Cat.

Consider the category $\text{Fun}(\mathcal{C}, \mathcal{A}) = \mathcal{J}$

$\text{Hom}_{\mathcal{J}}(F, G) = \{\text{natural transformations } \alpha: F \rightarrow G\}$

Given $\alpha, \beta: F \rightarrow G$ define $\alpha + \beta: F \rightarrow G$ via

$A \in \mathcal{C} \quad \alpha: F \rightarrow G$ $\alpha(A): F(A) \rightarrow G(A)$ $A \xrightarrow{f} B$ $F(A) \xrightarrow{\alpha(A)} G(A)$ $F(f) \downarrow$ $F(B) \xrightarrow{\alpha(B)} G(B)$	$(\alpha + \beta)(A) = \alpha(A) + \beta(A)$ <p>Show that this gives</p> <p>$\mathcal{J} = \text{Fun}(\mathcal{C}, \mathcal{A})$ the structure of an Ab-Cat.</p>
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Examples $\mathcal{C} = \bullet \quad \mathcal{A} = \text{Mod}_R$

$\mathcal{C} = \rightsquigarrow \quad A \xrightarrow{f} B$

Ex \star same w/ \mathcal{C} Ab-Cat $\mathcal{J} = \text{AddFun}(\mathcal{C}, \mathcal{A})$

Def An additive Category is an Ab-Cat \mathcal{A}
 s.t. • \exists a 0-object in \mathcal{A} (initial & final)
 • $A \times B$ exists for any $A, B \in \text{ob}(\mathcal{A})$

Examples $\text{Mod}_R, \text{SES}_R, \text{Fun}(\mathcal{C}, \mathcal{A}), \text{Fun}_{\text{Add}}^{\text{Ab}}(\mathcal{C}, \mathcal{A})$
 \mathcal{A} an add cat

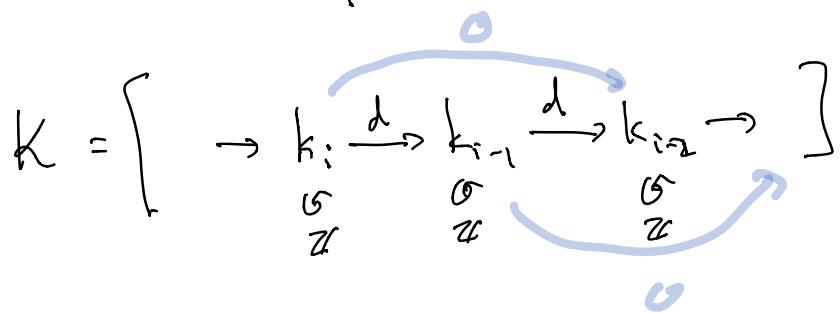
Def A chain complex in \mathcal{A} , \mathcal{A} is an Ab-Cat
 is a collection of objects $\{A_i\}_{i \in \mathbb{Z}} = A_{\bullet}$
 & morphisms $d_i: A_i \rightarrow A_{i-1}$ s.t.
 $d_{i+1}d_i = 0 \in \text{Hom}_{\mathcal{A}}(A_i, A_{i-2})$

Def A morphism of chain complexes $A_{\bullet} \xrightarrow{f_{\bullet}} B_{\bullet}$
 in \mathcal{A} is a sequence of morphisms $A_i \xrightarrow{f_i} B_i$
 such that f_i , the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B_i \\ d_i \downarrow & & \downarrow d_i \quad (\text{commutes}) \\ A_{i-1} & \xrightarrow{f_{i-1}} & B_{i-1} \end{array}$$

Note: if \mathcal{A} is an Ab-Cat then s_a is $\text{Ch}(\mathcal{A})$
 if \mathcal{A} is an Additive Cat s_a is $\text{Ch}(\mathcal{A})$ (component-wise)

$$\text{Ch}(\mathcal{A}) = \text{Fun}_{\text{Add}}(K, \mathcal{A})$$



Def Let \mathcal{A} be an additive category, $f: B \rightarrow C$

Then

- The kernel of f is a morphism $K \rightarrow B$
 such that $K \rightarrow B \xrightarrow{f} C$ and such that

$$\circlearrowleft$$

K is universal with this property in the
 sense that if $K' \rightarrow B$ is any morphism
 s.t. $K' \rightarrow B \xrightarrow{f} C$ then \exists unique $K' \rightarrow K$

$$\circlearrowleft$$

such that the diagram

$$\begin{array}{ccc} K' & \xrightarrow{\quad f \quad} & K \\ & \searrow & \downarrow \\ & & B \end{array} \text{ commutes.}$$

we write $K = \ker f$

$$\text{Alternately, } K = \lim_{\leftarrow} \left(\begin{array}{c} B \xrightarrow{f} C \\ \circ \end{array} \right)$$

- The cokernel of f is a morphism $C \rightarrow D$ s.t.

$$\text{s.t. the composition } \underbrace{B \xrightarrow{f} C \rightarrow D}_{C} \text{ is } \circ$$

and which is universal for this in the
sense that if $C \rightarrow D'$ w/ $\underbrace{B \rightarrow C \rightarrow D'}_{C}$

then $\exists!$ morphism $D \rightarrow D'$ s.t.

$$\underbrace{C \rightarrow D \rightarrow D'}_{C} \text{ commutes.}$$

$$\text{Alternately } D = \text{coker}(f) = \lim_{\rightarrow} \left(\begin{array}{c} B \xrightarrow{f} C \\ \circ \end{array} \right)$$

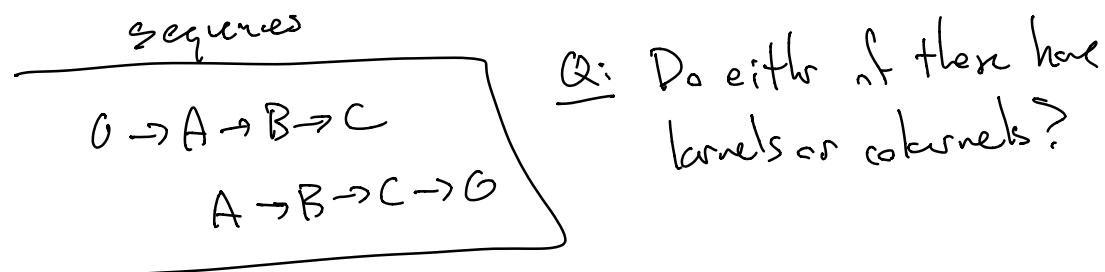
- We say that f is monic if for every $B' \xrightarrow{g} B$
w/ $\underbrace{B' \xrightarrow{g} B \rightarrow C}_{C}$ we have $g = 0$.

$$\underbrace{\quad}_{C} \qquad \qquad (\ker = 0)$$

- We say that f is epic if for any $C \xrightarrow{g} C'$
 w/ $B \xrightarrow{\quad} C \xrightarrow{g} C'$ we have $g = 0$
 $\xrightarrow{\quad}$ (coker = 0)

Exercise 8 Show that SES_R need not have kernels or cokernels.

- LES_R (RES_R) be cat. of left (right) exact short



Def An Abelian Category is an additive category A such that:

- every morphism has a kernel & a cokernel
- every monic is the kernel of its cokernel
- every epic is the cokernel of its kernel

$$A \xhookrightarrow{\text{monic}} B$$

$$B \rightarrow B/A$$

$$\ker(B \rightarrow B/A) = A$$

$$B \rightarrow B/A \quad \text{epic}$$

$$\text{kernel } A \rightarrow B$$

Prop. if A is an Ab. category so is $\text{Fun}_*(\mathcal{C}, A)$
and so is $\text{Ch}(A)$

Exercise if we consider $\text{SES}_{\mathbb{R}}$ as subcat of
 $\text{Ch}(\text{Mod}_{\mathbb{R}})$ then the smallest Ab^{category} subcat
of $\text{Ch}(\text{Mod}_{\mathbb{R}})$ containing $\text{SES}_{\mathbb{R}}$ & containing
all objects of $\text{Ch}(-)$ is a. to its objects
($\in \text{Ch}(-)$).