Sheaves of modules

Secret subtext: maps to projective space (section 7)

given some scheme X, can we realize $X \subset \mathbb{P}^n_A$ closed

same as $X \simeq \operatorname{Proj} S_\bullet$

where $S_\bullet$ graded alg, gen in deg 1 as an alg over S_0

$$S_\bullet = \frac{S_0[x_0, \dots, x_d]}{I}$$

I a hom ideal

gen. $x_1, \dots, x_d \in S_1$ $\qquad S_0 = A$

consider this as zero locus of I in $\mathbb{P}^d_A$

"Goal: find S_1"

i.e. linear forms $\longleftrightarrow$ deg 1 elements in coord alg. hom.

Insight: these form a coherent sheaf (locally free)

Strategy: identify these sheaves $\mathcal{O}_{\operatorname{Proj} S}(n)$ "locally" deg n forms

line bundles

Later: learn when a invertible sh $\mathcal{L}$ can be realized as an $\mathcal{O}_{\operatorname{Proj} S}(1)$

"very ample"

?

"ample"

?

"glob-gen"

<u>Recall:</u> S graded ring $S_\bullet = \bigoplus_{i=0}^{\infty} S_i$

can define $\operatorname{Proj} S$

for $f \in S_d$, $D^+(f)$ basic open in $\operatorname{Proj} S$
$$D^+(f) \cong \operatorname{spec} S_{(f)}$$

If M a graded module $M_\bullet = \bigoplus_{i=-\infty}^{\infty} M_i$

defined $\tilde{M}$ q. coherent sheaf

$$\tilde{M}\big|_{D^+(f)} \simeq \widetilde{M_{(f)}}$$

<u>Def</u> S graded ring, $n \in \mathbb{Z}$, define $\mathcal{O}_{\operatorname{Proj} S}(n) = \widetilde{S(n)}$

<u>Aside:</u> if M is a graded S-module,

$$M(n) = \bigoplus_i M(n)_i \qquad M(n)_i = M_{i+n}$$

$$\widetilde{S(n)}\big|_{D^+(f)} = \widetilde{S(n)_{(f)}}$$

$S(n)_{(f)}$ = elements of S_f hom of degree n

<u>Def</u> for $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-mods $X = \operatorname{Proj} S$

$$\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$$

$\mathcal{O}_X(1)$ = "Serre's twisting sheaf"

<u>Can check:</u> $\widetilde{M(n)} = \widetilde{M}(n)$ $\sim$

$\boxed{\text{Aside: if } X = \text{affine} \quad \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} = \widetilde{M \otimes_R N}}$

$$\widetilde{M}(n)\big|_{D^+(f)}$$
$$\shortparallel$$
$$\widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)\big|_{D^+(f)} = \widetilde{M}\big|_{D^+(f)} \otimes \mathcal{O}_X(n)\big|_{D^+(f)}$$
$$= \widetilde{M_{(f)} \otimes_S S(n)_{(f)}}$$
$$\downarrow$$
$$\widetilde{M(n)_{(f)}}$$

$$m \otimes s \mapsto ms$$

	-1	0	1	2
$S(1)$	S_0	S_1	S_2	S_3
M	… M_{-1}	M_0	M_1	M_2
$M(1)$	M_0	M_1	M_2	M_3

$M \otimes S(1) \to M(1)$ map of graded modules

Back up

if S graded ring, M, N graded modules
can form new graded module $M \otimes_S N$

$$(M \otimes_S N)_i = \sum_{a+b=i} M_a \otimes_{S_0} N_b$$

$$m \otimes sn = sm \otimes n$$

$a \quad c \quad b$

$$M_a \otimes N_{c+b} \qquad M_{a+c} \otimes N_b$$

$$M(n)_i = M_{n+i}$$

$$M(n) \otimes_S N(m) \simeq (M \otimes N)(n+m)$$

$$\widetilde{M \otimes_S N} = \tilde{M} \otimes_{\mathcal{O}_{\mathrm{Proj}\, S}} \tilde{N}$$

in particular $\quad \widetilde{M \otimes_S S(n)} = \widetilde{M(n)}$

$$\tilde{M} \otimes_{\mathcal{O}_{\mathrm{Proj}\, S}} \mathcal{O}_{\mathrm{Proj}\, S}(n) \equiv \tilde{M}(n)$$

$X = \mathrm{Proj}\, S$

- $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \simeq \mathcal{O}_X(n+m)$
- if S, T both graded rings, both gen. in deg 1 over S_0, T_0

and $\varphi : S \longrightarrow T$ is a graded ring hom

induced map $U \xrightarrow{f} \operatorname{Proj} S$ and $f^*\mathcal{O}(1) = \mathcal{O}(1)|_U$

$\operatorname{Proj} T > U$ open $U = \{ p \in \operatorname{Proj} T \mid \varphi(S_+) \not\subseteq p \}$

<u>Def</u> if $X = \operatorname{Proj} S$, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules

we define $\Gamma_* \mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \Gamma(\mathcal{F}(n))$

<u>Prop</u>: A a ring $S = A[x_0, \dots, x_n]$

$X = \operatorname{Proj} S = \mathbb{P}^n_A$ then $\Gamma_* \mathcal{O}_X = S$

this doesn't work for all graded S.

<u>Pf</u>: $\{ s \in \Gamma(\mathcal{O}_X(d)) \}$ in bijection w/

$$\{ (s_i)_{i=0,\dots,n} \mid s_i \in \Gamma(\mathcal{O}_X(n)|_{D^+(x_i)}) \text{ s.t. } s_i|_{D^+(x_ix_j)} = s_j|_{D^+(x_ix_j)} \}$$

$$\Gamma(\mathcal{O}_X(d)|_{D^+(x_i)})$$

$$\mathcal{O}_X(d)(D^+x_i) = S(d)_{(x_i)}$$

deg d in S_{x_i}

$s_0, s_1, \dots,$ $s_i \in S_{x_i}$ s_i deg d and $s_i = s_j$ in $S_{x_ix_j}$

$S_{x_i} \subset S_{x_ix_j}$ x_i are nonzero divisors

s_i's all elmts of $S_{x_0 x_1 \cdots x_n}$

we have $s = s_0 = s_1 \cdots$ in $S_{x_0 \cdots x_n}$

s.t. s in S_{x_0} & S_{x_1} ... deg d.

$s = x_0^{m_0} x_1^{m_1} \cdots x_n^{m_n} f(x_0, \ldots, x_n)$

uniquely

has no more factors of x_i's.

$\rightsquigarrow m_i = 0$ so $s =$ hom poly of deg d.

i.e. natural inclusion

$\Gamma(S(d)) \hookrightarrow S_{x_0 x_1 \cdots x_n}$

image is S_d

... eventually $\Gamma_*(\mathcal{O}_X) \simeq S$.

Very important fact

- $\mathcal{O}_X(d)$ is an invertible sheaf if S is gen in deg 1 as an S_0 algebra.

if $f_1, \ldots, f_m \in S_1$ generators,
cover Proj $S_\bullet$ w/ $D^+(f_i)$

$M \rightsquigarrow \tilde{M}$

$\mathcal{F} \to \Gamma_* \mathcal{F}$

Key proposition (5.15)

if $S_\bullet$ f.g. in deg 1 over S_0, $X = \operatorname{Proj} S_\bullet$
and $\mathcal{F}$ q.coh on X then $\exists\, \beta : \widetilde{\Gamma_* \mathcal{F}} \xrightarrow{\sim} \mathcal{F}$

$$S \rightsquigarrow \tilde{S} = \mathcal{O}_{\operatorname{Proj} S} \longrightarrow \Gamma_*(\mathcal{O}_{\operatorname{Proj} S})$$

S poly $\Rightarrow$ $\overset{\shortparallel}{S}$

Definition: recall $\mathbb{P}^n_X = X \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$

$$\mathcal{O}_{\mathbb{P}^n_X}(d) = \pi_2^* \mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(d)$$

when is X a relative proj variety?

i.e.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \mathbb{P}^n_Y \arrow[d] \\ & Y \end{tikzcd}$$

Given a Y-scheme X, we say an invertible sheaf $\mathcal{L}$ on X is very ample relative to Y if $\exists$ an immersion $X \xrightarrow{i} \mathbb{P}^n_Y$ s.t. $i^* \mathcal{O}_{\mathbb{P}^n_Y}(1) \simeq \mathcal{L}$

<u>Def</u> a sheaf $\mathcal{F}$ of $\mathcal{O}_X$ modules on X is gen. by global sections if $\exists\ s_i \in \Gamma(\mathcal{F})$, $i \in I$ s.t. $\forall x \in X$ the images of s_i generate stalk $\mathcal{F}_x$ over $\mathcal{O}_{X,x}$

<u>Thm</u> X proj over noeth ring A i.e. $X \xhookrightarrow{i} \mathbb{P}^n_A$ and $\mathcal{L} = i^*\mathcal{O}(1)$

then for $\mathcal{F}$ coherent $\mathcal{O}_X$ mod, $\exists\ n_0 >> 0$ s.t. $\mathcal{F} \otimes \mathcal{L}^n = \mathcal{F}(n)$ is generated by a finite # of global sections for all $n \geq n_0$

<u>Thm</u> if $X \xrightarrow{f} Y$ projective (i.e. $X \hookrightarrow \mathbb{P}^n_Y \to Y$, closed)

then f_*(coherent) $=$ coherent.