Lecture 4: sites, topoi and cohomology

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Last time: **Groth. top** on a category $\mathcal{C}$

$$X \rightsquigarrow \mathrm{Cov}(X) = \{ \{U_i \to X\} \}$$

(need fiber products)

Site: Cat + Grotl. top

Ex:

- X top space: $\mathrm{Op}(X)$ → objects open $U \subset X$
  (tiny) → $\mathrm{Mor}(U,V) = \begin{cases} \emptyset & U \not\subset V \\ \{*\} & U \subset V \end{cases}$

$$U_1, U_2 \subset U \qquad U_1 \times_U U_2 = U_1 \cap U_2$$

$$\mathrm{Cov}(U) = \{\text{covs}\}$$

- X top space: $\mathcal{C}_X$ → objects morphisms $U \to X$
  (small) top spaces
  identifying $U \simeq$ Open in X

covs are things that identify w/ covs (actual) $\rightsquigarrow$ morph

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

comm. diagrams.

- X top space $\mathcal{C}$ = All morphisms $U \to X$
  (big)
  $\mathrm{Cov}\left(\begin{smallmatrix} U \\ \downarrow \\ X \end{smallmatrix}\right)$ are $\{U_i \to U\}$ s.t. $\{U_i \to U\}$ a cov in $\mathcal{C}_U$

X scheme

- Zariski$/X$ (small top site for X as a top space)
  $\mathrm{Op}(X)$

- Big Zariski$/X$ $\quad \mathcal{C} = \mathrm{Sch}/X$ $\quad$ covs:

$$\mathrm{Cov}(U) = \{\{U_i \to U\} \mid U_i \to U \text{ open embedding}, \ \textstyle\coprod U_i \to U \text{ surjective}\}$$

Ex: Étale

$\text{Ét}(X)$ = subcategory of Sch/X consisting of $(U \to X)$

$f: U \to X$ f étale. $\quad \mathrm{Cov}(U) = \{\{U_i \to U\} \mid \coprod U_i \to U \text{ surjective}\}$

Ex: Big Étale

$\mathcal{C} = \mathrm{Sch}/X \quad \mathrm{Cov}(U) = \{\{U_i \to U\} \mid \text{it's a cov in } \text{Ét}(U)\}$

Ex: $\mathrm{fppf}/X = \mathrm{fffp}/X \qquad \mathcal{C} = \mathrm{Sch}/X$

$\mathrm{Cov}(U) = \{\{U_i \to U\} \mid U_i \to U$ is locally finitely presented flat, $\coprod U_i \to U$ surjective$\}$

Ex: Lisse-ét $\quad \mathcal{C} = \mathrm{Sm}/X$, $\mathrm{Cov}(U)$ = same as big Étale$/X$

[illegible] top $\quad \mathcal{C} = \mathrm{Sm}/X \quad$ covs from fppf.

(formally smooth + loc. f. pres. = smooth)

Making new sites from old

- $\mathcal{C}$ cat, $\rightsquigarrow \mathcal{C}/X$ $\quad$ $\mathrm{Ob}(\mathcal{C}/X) = \{(Y,f) \mid f: Y \to X\}$

$\mathrm{Mor}((Y,f),(Z,g))$

$$\begin{tikzcd} Y \arrow[r] \arrow[dr, "f"'] & Z \arrow[d, "g"] \\ & X \end{tikzcd}$$

- If we have a Groth top on $\mathcal{C}$ get one on $\mathcal{C}/X$

via: $\mathrm{Cov}(Y,f) = \{ \{(U_i, g_i) \to (Y,f)\} \mid \{U_i \to Y\} \in \mathrm{Cov}(Y) \}$

- $\mathcal{C}$ cat, Δ another cat $\quad F: \Delta^{op} \to \mathcal{C}$

"Diagram in $\mathcal{C}$"

$\mathcal{C}/F$ $\quad$ $\mathrm{Ob}(\mathcal{C}/F) = \{(Y, \delta, \varphi) \mid \varphi: Y \to F(\delta)\}$

$\mathrm{Mor}((Y,\delta,\varphi),(Z,\eta,\psi))$

$f: \eta \to \delta \quad f^b: Y \to Z$

c^h

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "\varphi"'] & Z \arrow[d, "\psi"] \\
F(\delta) \arrow[r, "F(f)"'] & F(\eta)
\end{tikzcd}$$

<u>Def</u> A presheaf (of sets) on a cat $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \longrightarrow \underline{Set}$

$\hat{\mathcal{C}}$ = Cat of presheaves

$\mathcal{C}$ a site, can

<u>Define</u> $F \in \hat{\mathcal{C}}$ is separated if $\forall U \in \mathcal{C}$

$\{U_i \to U\} \in Cov(U)$, $F(U) \hookrightarrow \prod_i F(U_i)$

<u>Def</u> $F \in \hat{\mathcal{C}}$ is a sheaf if $\forall U \in \mathcal{C}$

$\{U_i \to U\} \in Cov(U)$

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

is exact $\qquad (\alpha_i) \rightrightarrows (\beta_{ij})$

$\hookrightarrow \beta_{ij} = F(U_i \times_U U_j \to U_i)(\alpha_i)$

$\hookrightarrow \beta_{ij} = F(U_i \times_U U_j \to U_j)(\alpha_j)$

"Recall" if A, B, C sets

Recall if A, B, C sets

$$A \xrightarrow{\alpha} B \overset{\beta}{\underset{\gamma}{\rightrightarrows}} C \quad \text{is exact if}$$

α is a bijection between A & $\mathrm{Eq}(B \overset{\beta}{\underset{\gamma}{\rightrightarrows}} C)$

$$\text{''} \quad \{ b \in B \mid \beta(b) = \gamma(b) \}$$

Thm Sheafification exists

ie, have a functor $\underline{\mathrm{Pre}\,\mathcal{C}} \longrightarrow \underline{\mathrm{Shv}\,\mathcal{C}} \; (\overset{\text{full subcat}}{\hookrightarrow} \underline{\mathrm{Pre}\,\mathcal{C}})$

$$F \longmapsto F^a$$

s.t. $\mathrm{Hom}(\underset{\overset{\nearrow}{\text{presheaf}}}{F}, \underset{\overset{\uparrow}{\text{sheaf}}}{G}) = \mathrm{Hom}(F^a, G)$

$$\begin{tikzcd}
F \arrow[r] \arrow[dr, dashed, "\exists!"'] & G \\
& F^a \arrow[u, dashed, "\exists!"']
\end{tikzcd}$$

In fact can "seperatify" presheaves

$F \longrightarrow F^s \qquad \mathrm{Hom}(F, \underset{\overset{\uparrow}{\text{sep}}}{G}) = \mathrm{Hom}(F^s, G)$

Usual Dance:

- $F^s(U) = F(U)/\sim$ where $\alpha \sim \beta$ if $\exists \{U_i \xrightarrow{f_i} U\} \in \mathrm{Cov}(U)$

  s.t. $F(U_i \to U)(\alpha) = F(U_i \to U)\beta$

  $f_i^* \alpha = f_i^* \beta$

- F sep, set

  $F^a(U) = \{ (\{U_i \to U\}, \{\alpha_i\}) \mid \{U_i \to U\} \in \mathrm{Cov}(U)$, $\alpha_i \in F(U_i)$,

$$F^{u}(U) = \left\{ \left(\{U_i \to U\}, \{\alpha_i\} \right) \mid \quad \alpha_i \in F(U_i), \quad \{\alpha_i\}_{i \in I} \in \mathrm{Eq}\left(\prod F(U_i) \rightrightarrows \prod F(U_i \times_U U_j) \right) \right\}$$

$\sim$ via refinements.

Def A topos (Groth. top) is a category T, equivalent to the category $Shv(\mathcal{C})$ of sheaves on some site $\mathcal{C}$.

Notation X scheme, $X_{ét}$, $X_{lis\text{-}ét}$, X_{fppf}, $X_{Ét}$

small the topoi big

Ex: $Et^{aff}(X)$ = subcat of $Et(X)$ if $U \to X$ U affine étale

inherits Groth. top.

Exercise (2.6) topoi of $Et^{aff}(X)$'s, $Et(X)$ are equiv.

Topoi "feel" like sets:

A, B sheaves, $A \times B$ sheaf

Sheaf of gps: sheaf G,
$m : G \times G \longrightarrow G$
$i : G \longrightarrow G$
$e : \{*\} \longrightarrow G$

$m \times id$

$e:\{*\} \longrightarrow G$

s.t.
$$\begin{tikzcd}
G \times G \times G \arrow[r, "m \times id"] \arrow[d, "id \times m"'] \arrow[dr, phantom, "\circlearrowright"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

Ab Sheaves
sheaves of rgs

= group, Ab gp, ring objects in T