Lecture 17: constructing algebraic spaces as quotients

Tuesday, October 14, 2014 11:05 AM

Today: Suppose X scheme$/S$, $R \hookrightarrow X \times_S X$ an étale eq. rel. $Y \equiv X/R$

Then: $\Delta : Y \to Y \times Y$ is representable.

($\Rightarrow$ from yesterday/earlier today that Y is an algebraic space).

let $j : U \to X$ open subscheme. let R_U be as in

$$\begin{tikzcd} R_U \arrow[r] \arrow[d] & U \times U \arrow[d] \\ R \arrow[r] & X \times X \end{tikzcd}$$

then $R_U \to U$ is étale & $\bar{j} : U/R_U \to X/R \cong Y$ is open (rep.) immersion

$$\begin{tikzcd} R_U \arrow[r] \arrow[d] \arrow[rr, bend left] & U \times U \arrow[r] \arrow[d] & U \arrow[d] \\ R \arrow[r] \arrow[rr, bend right] & X \times X \arrow[r] & X \end{tikzcd}$$

pullback of rep, ét is rep, ét $\Rightarrow R_U \to U$ ét

$\bar{j}$ is a monomorphism ✓

to see $\bar{j}$ open, consider $f : T \to Y$

$$\bigcirc \xrightarrow{\text{open}} T$$

$$
\begin{tikzcd}
\mathcal{O} \arrow[r, "\text{open}"] \arrow[d] & T \arrow[d, "f"] \\
U/R_U \arrow[r, "\bar{j}"'] & Y = X/R
\end{tikzcd}
$$

to see that $\mathcal{O} \to T$ open, can check

étale locally on T $\Rightarrow$ WLOG can assume

$$
\begin{tikzcd}
T \arrow[r, "f"] \arrow[dr, "\tilde{f}"'] & Y \\
& X \arrow[u]
\end{tikzcd}
$$

$T \xrightarrow{f} Y$ factors through X.

but now

thinking about eq. rels set theoretically

$$
\begin{tikzcd}
\mathcal{O} \arrow[r] \arrow[d] \arrow[dd, bend right] & T \arrow[d, "\tilde{f}"] \arrow[dd, bend left] \\
U \arrow[r] \arrow[d] & X \arrow[d] \\
U/R_U \arrow[r] & Y
\end{tikzcd}
$$

$\tilde{f}^{-1}(s(t^{-1}U))$ is open (for top. reasons in T.)

$$
\begin{tikzcd}
R \arrow[r] \arrow[rr, bend left, "s"] \arrow[rr, bend right, "t"'] & X \times_Y X \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X
\end{tikzcd}
$$

Let's start to look more directly at rep. of Δ_Y.

$$
\begin{tikzcd}
F \arrow[r] \arrow[d] & Y \arrow[d] \\
W \arrow[r] & Y \times Y
\end{tikzcd}
$$

want to show F is a scheme.

"basic strategy:"

$$\begin{tikzcd} F' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & F \arrow[d] \\ W' \arrow[r] & W \end{tikzcd}$$

$W' \to W$ étale

W, W' affine.

$F' \to W'$

q. affine.

$\rightsquigarrow$ ét. descent of q. affine morphisms that $F \to W$ is given by q. affine morph. of schemes.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & Y \arrow[d] \\ W \arrow[r] & Y \times Y \end{tikzcd}$$

can work Zariski locally on S $\Rightarrow$ can assume S affine.

can work Zar. locally on W $\Rightarrow$ can assume W affine.

$X \to Y$ et. surj. of sheaves

so is $X \times X \to Y \times Y$ $\Rightarrow$ can find $W' \to W$ étale cover s.t.

$$\begin{tikzcd} W' \arrow[r] \arrow[dr, bend right] & W \arrow[r] & Y \times Y \\ & X \times X \arrow[ur] & \end{tikzcd}$$

moreover, can assume that ...

Moreover, can assume ...

$$\begin{tikzcd} W \arrow[dr] & \cdots \\ & X \times X \arrow[ur] \end{tikzcd}$$

W' affine since W is affine &, hence q-compact.

note we have:

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \times X \arrow[d] \\ Y \arrow[r] & Y \times Y \end{tikzcd}$$

&, so

$$\begin{tikzcd}[row sep=small, column sep=small]
& F' \arrow[rr] \arrow[dd] \arrow[dl] \arrow[ddrr, phantom, "\ulcorner", very near start] & & W' \arrow[dd] \arrow[dl] \\
R \arrow[rr, crossing over] \arrow[dd] & & X \times X & \\
& F \arrow[rr] \arrow[dl] & & W \arrow[dl] \\
Y \arrow[rr] & & Y \times Y \arrow[from=uu, crossing over] &
\end{tikzcd}$$

cart

$\Rightarrow$

$F' \to F$ rep &, ét surj

(pullback of rep)

$\Rightarrow R' = F' \times_F F'$

this is an étale eq. rel

&, $F'/R' = F$

via fiber product, we see F' is a scheme.

&, $F' \longrightarrow W'$ a monomorphism. $\Rightarrow F'$ separated

because W' is sep. (affine)

etcetera. $\square$