ADSP Assignment-6 Question-2

EE18BTECH11014

1 Laplace Transform

Laplace Transform transforms a function of a real variable t (often time) to a function of a complex variable s (complex frequency). It is defined for function $f(t)$ where $t \geq 0$.

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

$$s = \sigma + j\omega$$

2 Fourier Transform

Fourier transform decomposes a function into its constituent frequencies. Fourier transform is defined for signals defined on $(-\infty, \infty)$.

$$\begin{aligned} H(\omega) &= \mathcal{F}[h(t)] \\ &= \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt \end{aligned}$$

3 Exploring Fourier Transform and Laplace Transform

Fourier Transform decomposes frequency components of Signal but not the Exponential Part of Signals. Whereas, Laplace Transform estimates both Frequency and Exponential Components of Signals.

Fourier transforms map a function to a new function on the real line, whereas Laplace Transform maps a function to a new function on the complex plane. We need 3 dimensions to represent Fourier Transform and 4 dimensions to represent Laplace Transform as in Laplace Transform we need to vary both σ and ω but in case of Fourier Transform we only vary ω.

Derivation of Fourier Transform form Laplace Transform

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

$$s = \sigma + j\omega$$

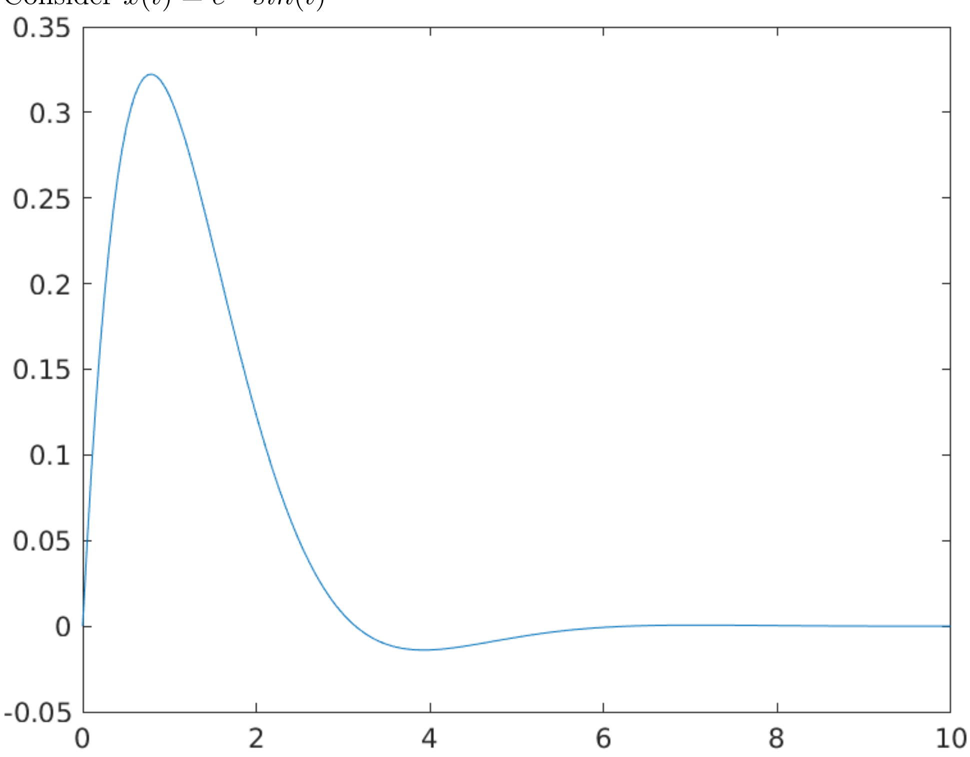
$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-(\sigma+j\omega)t}dt$$

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}dt$$

Laplace Transform of $x(t)$ is the Fourier Transform of $x(t)e^{-\sigma t}$. Laplace Transform evaluated at $s = j\omega$ is equal to the Fourier transform if its Region of Convergence (ROC) contains the imaginary axis. Fourier Transform is a part of Laplace Transform. Every function that has a Fourier transform will have a Laplace transform but not vice-versa. It can be illustrated by an example.

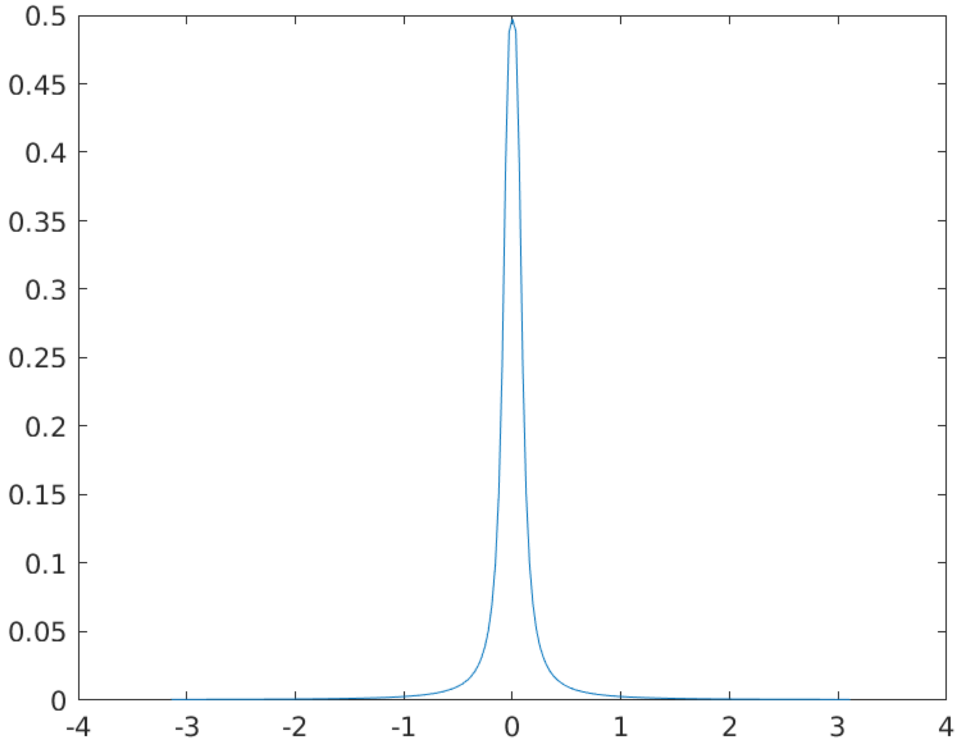
Example:

Consider $x(t) = e^{-t}sin(t)$



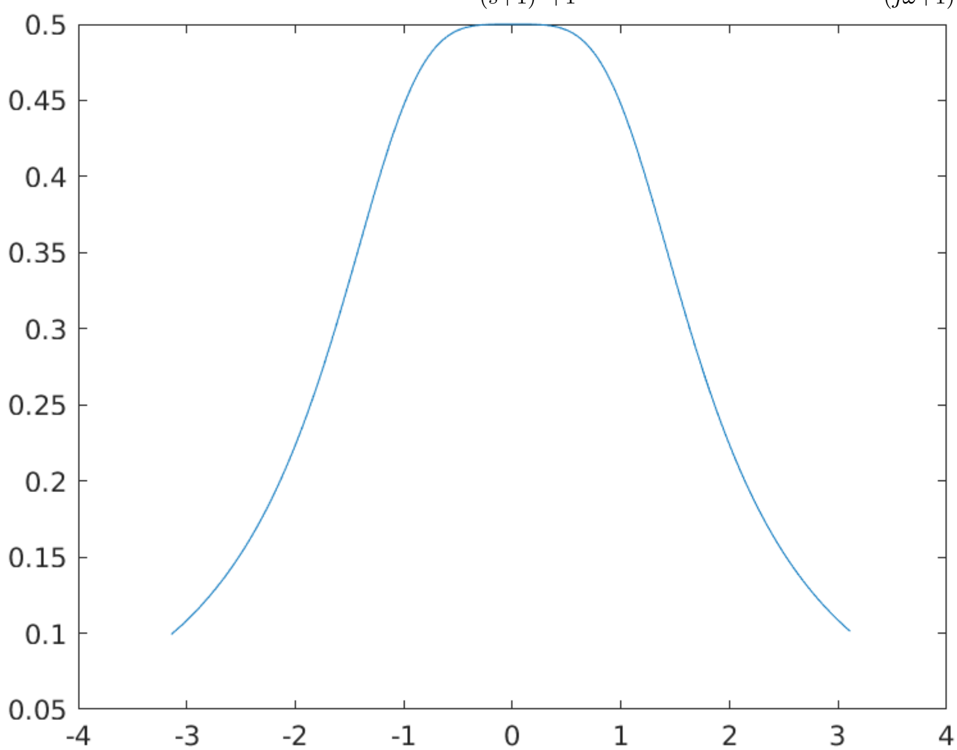
The Image is generated in Matlab by plotting the function.

Fourier Transform of $x(t)$ is $\frac{1}{1+(1+j\omega)^2}$



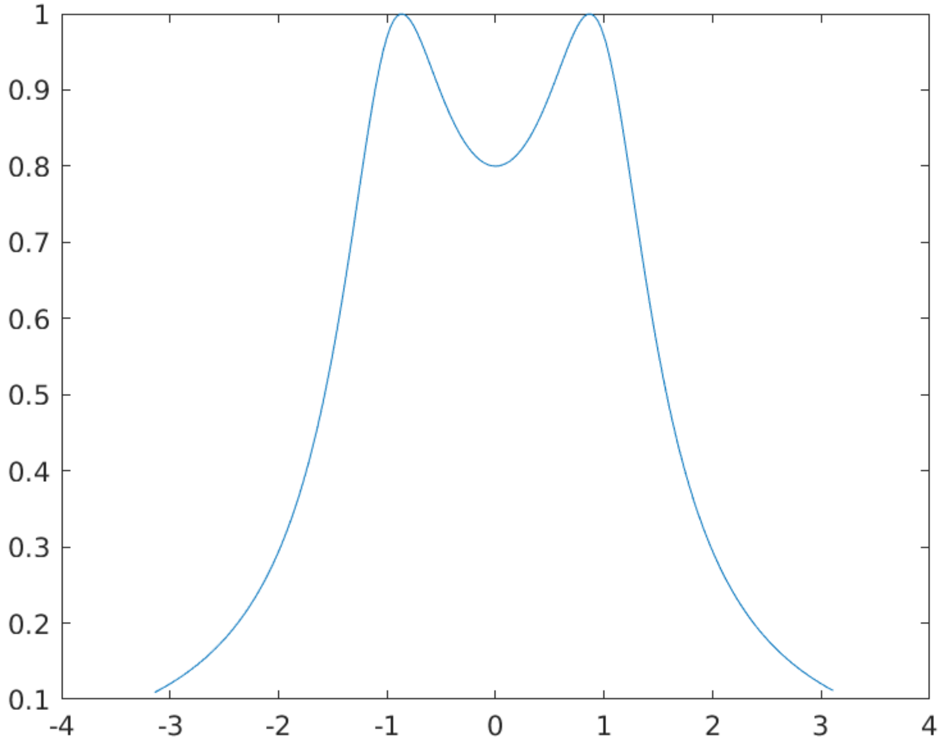
The Image is generated in Matlab by using "fft".

Laplace Transform of $x(t)$ is $X(s) = \frac{1}{(s+1)^2+1}$ If $s = j\omega$ then $X(s) = \frac{1}{(j\omega+1)^2+1}$



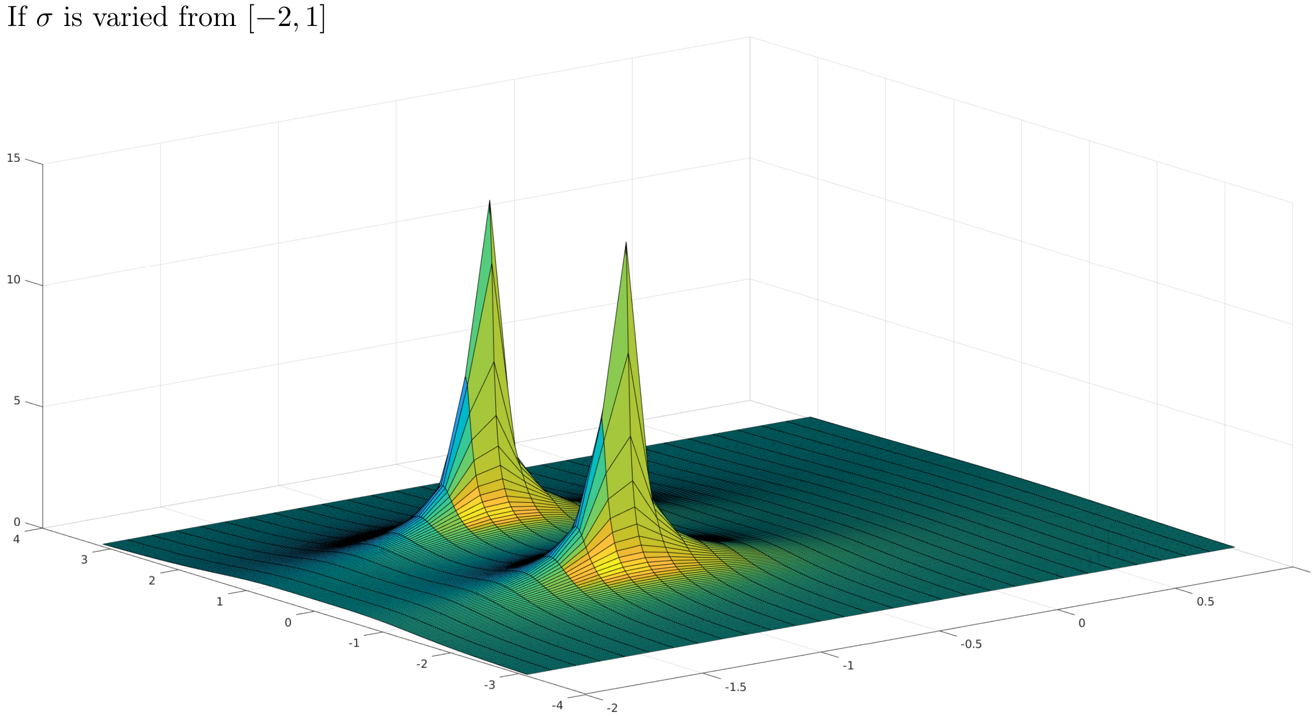
The Image is generated in Matlab by using "laplace" and applying $s = j\omega$ in Laplace Domain.

If $s = -0.5 + j\omega$ then $X(s) = \frac{1}{(j\omega+0.5)^2+1}$



The Image is generated in Matlab by using "laplace" and applying $s = -0.5 + j\omega$ in Laplace Domain.

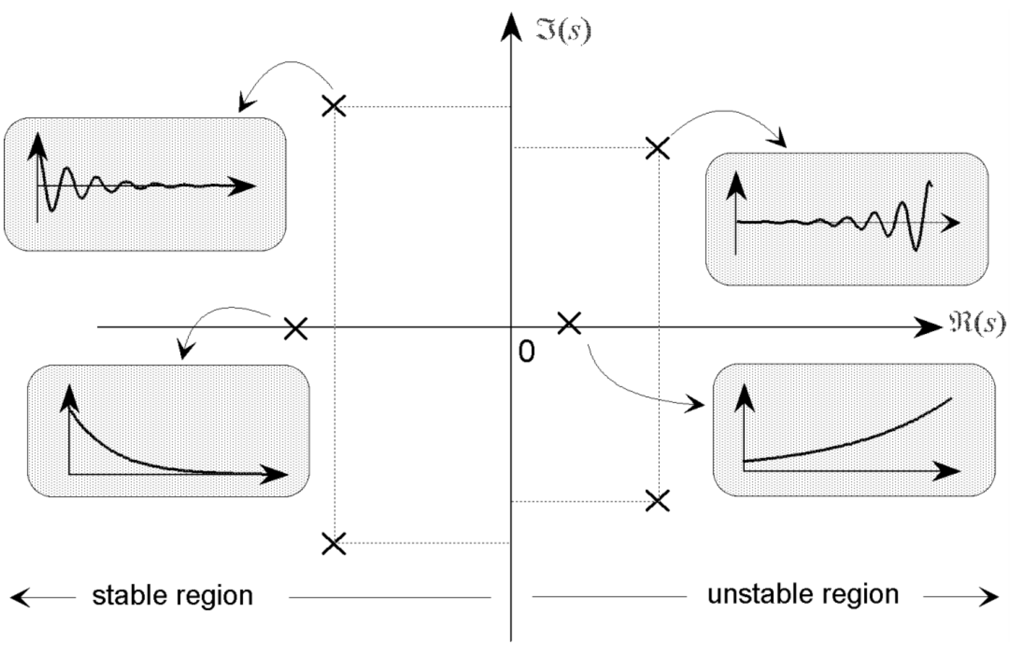
If σ is varied from $[-2, 1]$



Region of Convergence is the region of s-plane where the integral(of Laplace Transform) converges by definition. For the Transfer Function $X(s) = \frac{1}{(s+1)^2+1}$ the Region of Convergence(ROC) is Re(s) > -1. The peaks in the graph indicate the Poles of the Transfer Function. Location of Poles are $-1+j$ and $-1-j$.

4 Characterising Signal depending on Location of Poles

Consider the Transfer Function $H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ where the Poles are located at $\frac{-2\zeta\omega_n \pm 2\omega_n\sqrt{\zeta^2-1}}{2} = \sigma \pm j\omega_d$ where ζ is the Damping Factor. Depending on locating of Poles, the Behaviour of Signals changes. Signals are bounded if Poles of their Laplace Transform lie on left half of Imaginary Axis.

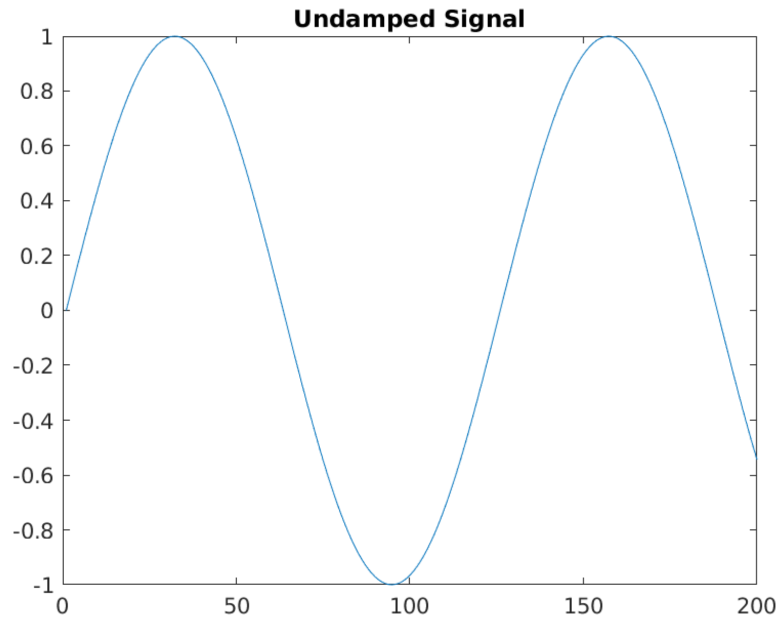


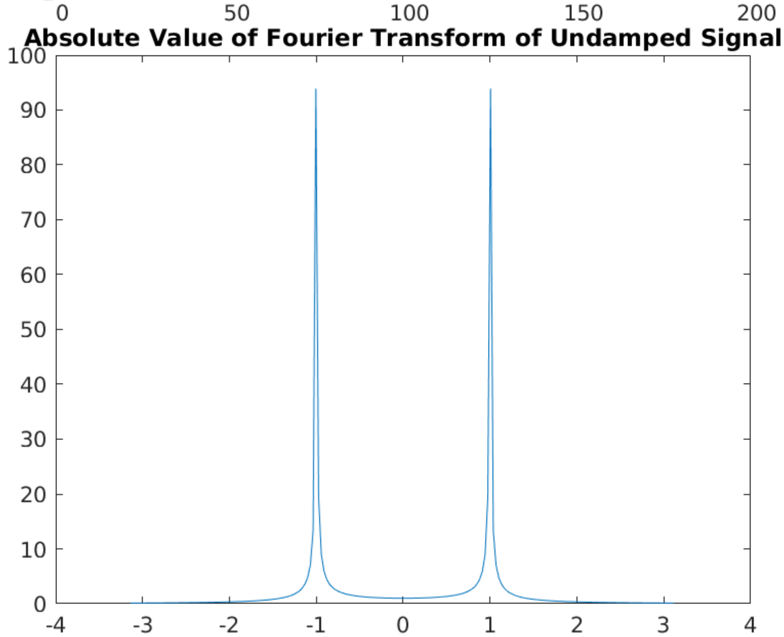
Signals are classified as:

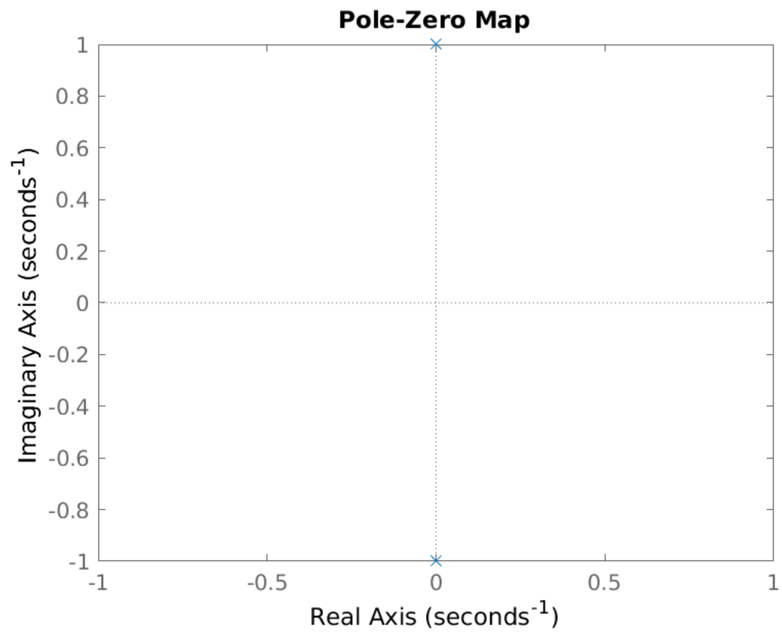
- Critically Damped Signals
- Damped Signals
- Undamped Signals
- Under Damped Signals

4.1 Undamped Signals

For this family of Signals $\zeta = 0$. i.e $\sigma = 0$, which implies that poles lie on Imaginary Axis. Consider the signal $x(t) = sin(t)$. Laplace Transform of $x(t)$ is $X(s) = \frac{1}{s^2+1}$. So Poles of Transfer Function are $-j, j$.



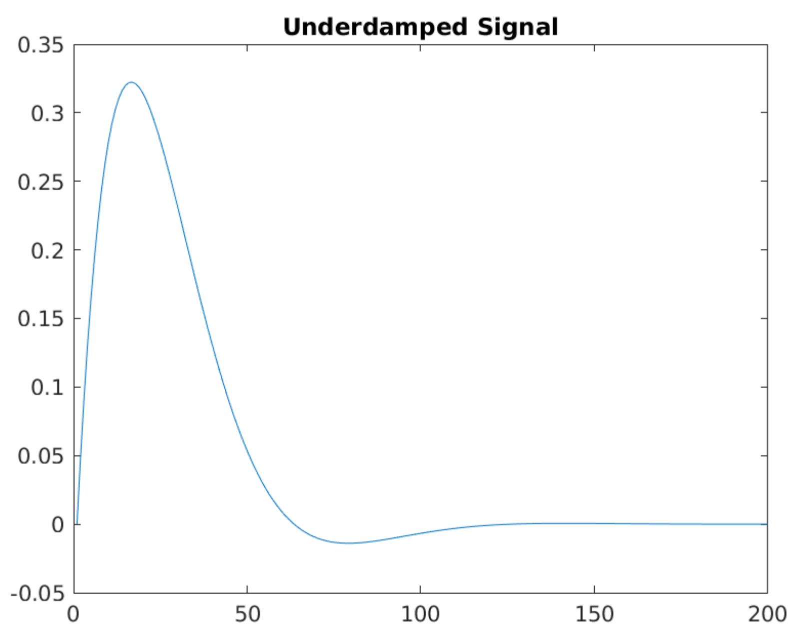


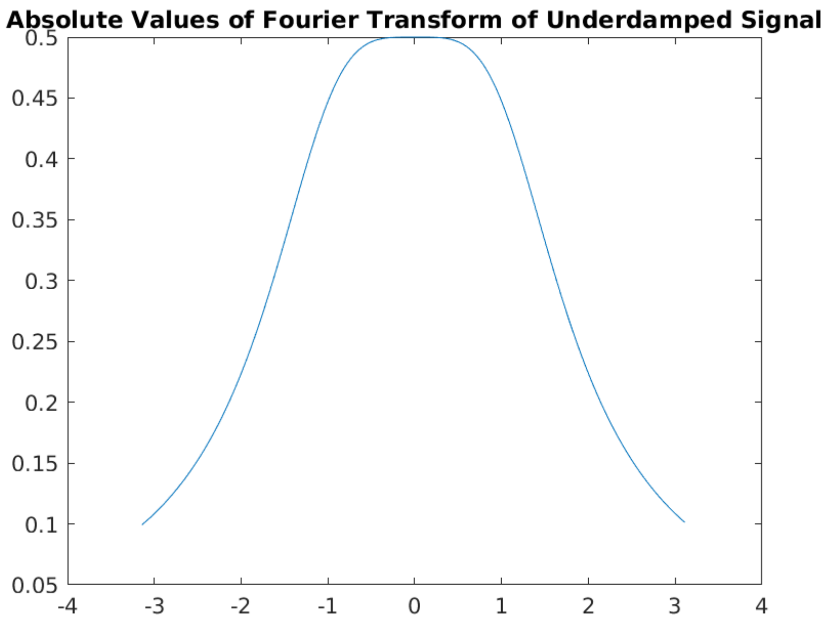


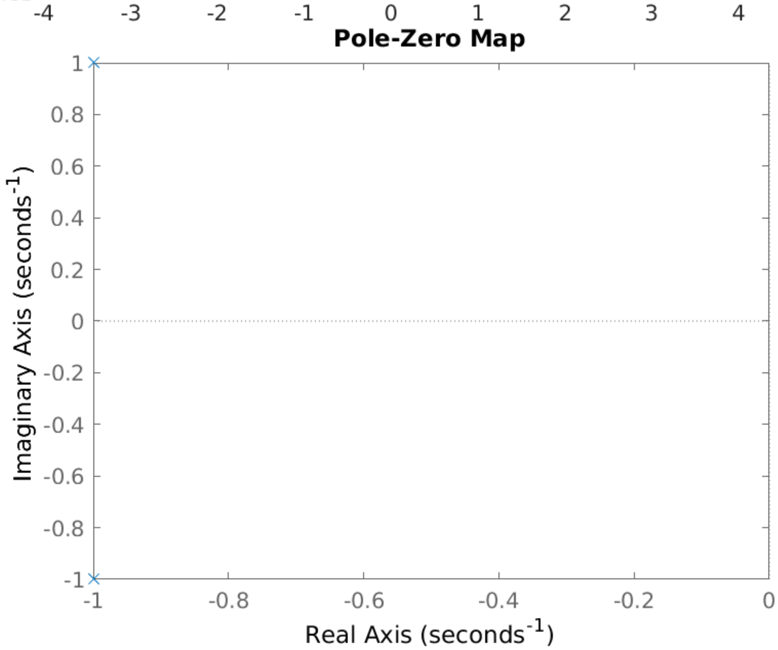
4.2 Underdamped Signals

For this family of Signals $0 \leq \zeta < 1$. It implies that poles lie on Complex Conjugates of each other.

Consider the signal $x(t) = e^{-t}sin(t)$. Laplace Transform of $x(t)$ is $X(s) = \frac{1}{(s+1)^2+1}$. So Poles of Transfer Function are $-1-j, -1+j$.

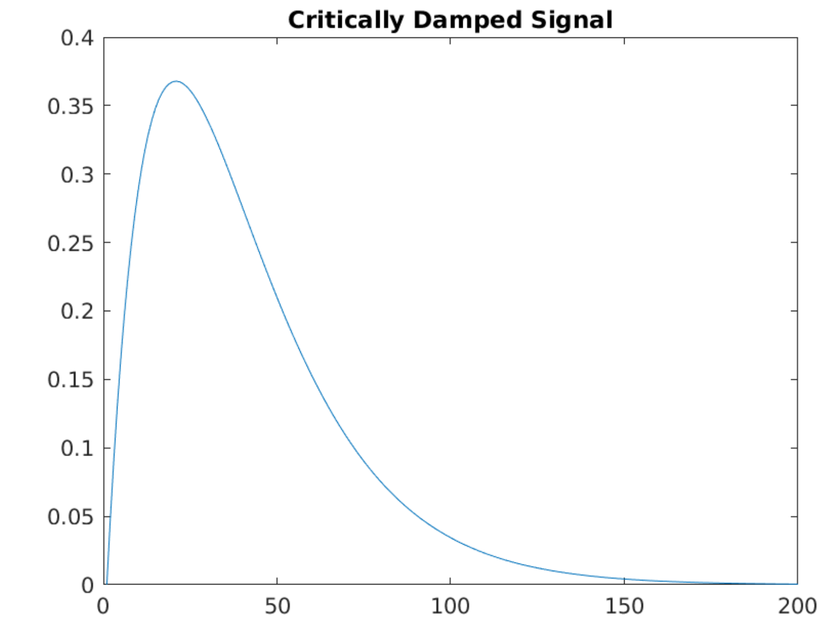


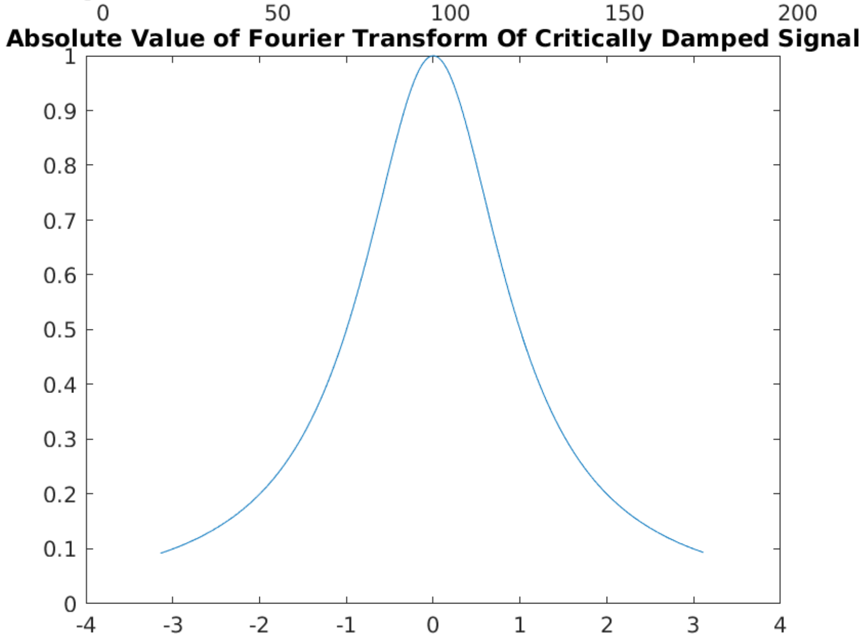


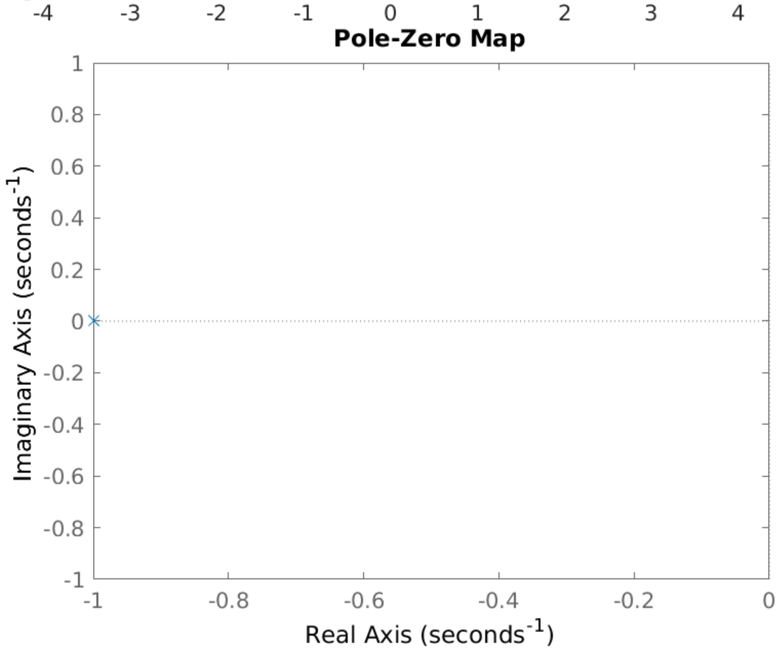


4.3 Critically Damped Signals

For this family of Signals $\zeta = 1$. It implies that poles are real and are equal each other. Consider the signal $x(t) = te^{-t}$. Laplace Transform of $x(t)$ is $X(s) = \frac{1}{(s+1)^2}$. So Poles of Transfer Function are -1.

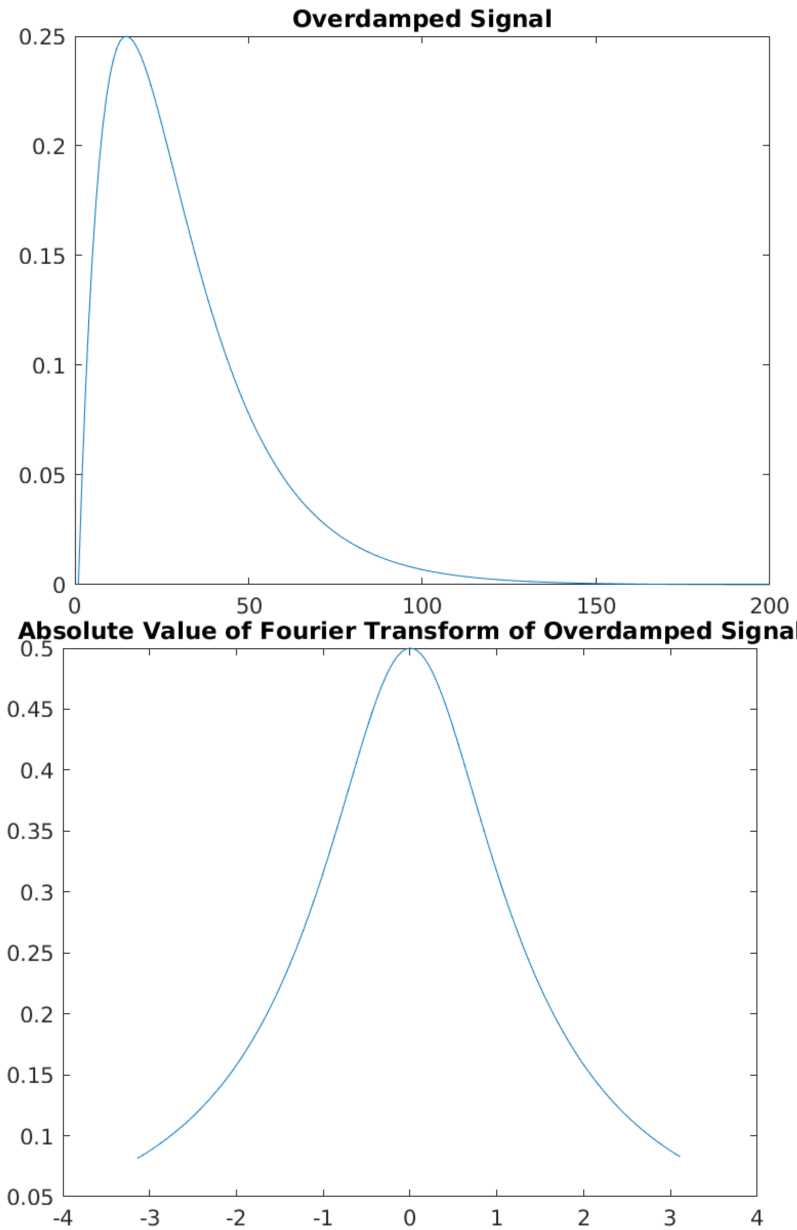

Critically Damped Signal
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
50
100
150
200

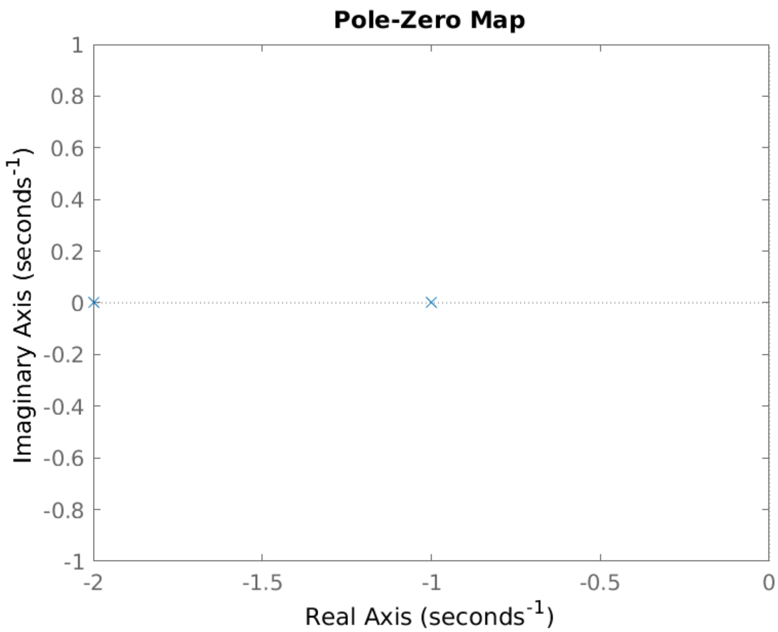

Absolute Value of Fourier Transform Of Critically Damped Signal
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-4
-3
-2
-1
0
1
2
3
4


Pole-Zero Map
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0

4.4 Overdamped Signals

For this family of Signals $\zeta > 1$. It implies that poles are real and distinct. Consider the signal $x(t) = e^{-t} - e^{-2t}$. Laplace Transform of $x(t)$ is $X(s) = \frac{1}{(s+1)(s+2)}$. So Poles of Transfer Function are $-1, -2$.





4.5 Observations

It can be observed that Fourier Transform doesn't provide us details about the Exponential term in $x(t)$ while Laplace transform provides us about both Frequency Components and Exponential Terms.

4.6 Root Locus Plot

Root Locus Analysis is a graphical method for examining how the roots of a system change with variation of a certain system parameters.

For example by varying ζ and ω_n in Transfer Function $H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

5 Acknowledgements

https://www.youtube.com/watch?v=n2y7n6jw5d0

https://web.mit.edu/2.14/www/Handouts/PoleZero.pdf