# Control Systems

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Contents



***Abstract*—This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/codes
```

## 1 Circuit Design

1.1. Consider the Magnitude Bode Plot and Phase Bode Plot 1.1 of Open-Loop Transfer Function of an Amplifier. Estimate the Open-Loop Transfer Function. (Assume $'A'$ as $'G'$ and $'\beta'$ as $'H'$)

**Solution:** Let $G(f)$ be the Open-Loop Transfer Function,

$$G(f) = \begin{cases} 100 & 0 < f < 10^5 \\ 200 - 20\log(f) & 10^5 < f < 10^6 \\ 320 - 40\log(f) & 10^6 < f < 10^7 \\ 460 - 60\log(f) & 10^7 < f \end{cases} \tag{1.1.1}$$

$$\nabla G(f) = \frac{d(G(f))}{d(\log(f))} = \begin{cases} 0 & 0 < f < 10^5 \\ -20 & 10^5 < f < 10^6 \\ -40 & 10^6 < f < 10^7 \\ -60 & 10^7 < f \end{cases} \tag{1.1.2}$$

As we know that, **When a pole is encountered the slope always decreases by 20 dB/decade** and **When a zero is encountered the slope always increases by 20 dB/decade**. So, by observing Fig. 1.1 it can be concluded that we are having Poles at $f = 10^5 Hz, 10^6 Hz, 10^7 Hz$ and No Zeros.

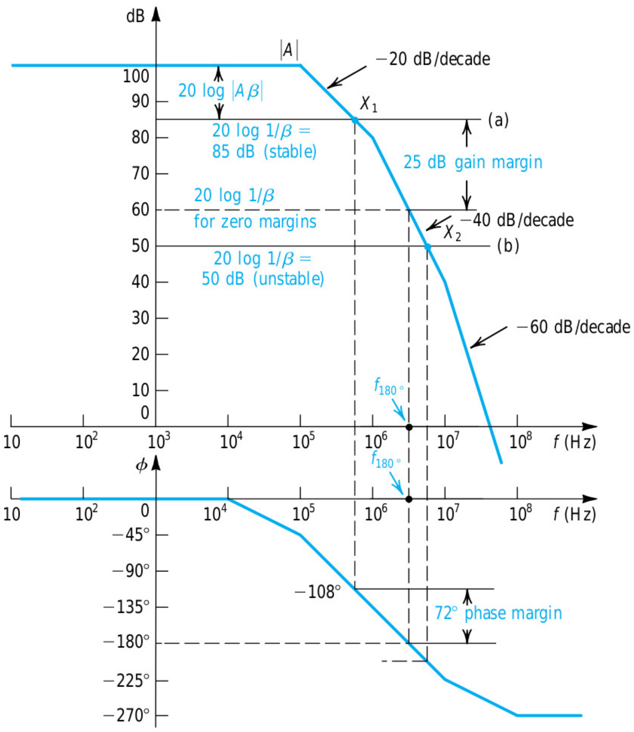


Fig. 1.1: Magnitude and Phase Bode Plot

So, the Open-Loop Transfer Function $G(f)$ is

$$G(f) = \frac{10^5}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \tag{1.1.3}$$

1.2. Calculate the Phase of Open-Loop Transfer Function.

**Solution:**

$$\phi(f) = -\left[\tan^{-1}\left(\frac{f}{10^5}\right) + \tan^{-1}\left(\frac{f}{10^6}\right) + \tan^{-1}\left(\frac{f}{10^7}\right)\right] \tag{1.2.1}$$

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1.3. Find the PM from Fig. 1.1, given that he feedback gain $H(f)$ is constant and given by

$$20\log\left(\frac{1}{H(f)}\right) = 85dB \tag{1.3.1}$$
$$\text{or, } H(f) = 5.623\times 10^{-5}. \tag{1.3.2}$$

**Solution:** From the figure,

$$20\log|G(f_1)| = 85dB \tag{1.3.3}$$
$$\implies 20\log|G(f_1)| = 20\log\left(\frac{1}{H(f_1)}\right) \tag{1.3.4}$$
$$\text{or, } |G(f_1)H(f_1)| = 1 \tag{1.3.5}$$

and

$$f_1 = 0.493MHz, \tag{1.3.6}$$

from (1.4.4) and (1.1.3). Also,

$$\because \underline{/H(f)} = 0, \forall f \tag{1.3.7}$$
$$\underline{/G(f_1)H(f_1)} = \underline{/G(f_1)} = -108^\circ \tag{1.3.8}$$
$$\implies PM = 180^\circ - 108^\circ = 72^\circ \tag{1.3.9}$$

using (1.3.6) in (1.2.1).

1.4. Find the GM.
**Solution:** The crossover frequency $f_\pi$ is defined as

$$\underline{/G(f_\pi)H(f_\pi)} = 180^\circ \tag{1.4.1}$$
$$\implies \underline{/G(f_\pi)} = 180^\circ \tag{1.4.2}$$
$$\implies f_\pi = 3.34MHz \tag{1.4.3}$$

by solving (1.2.1). From Fig. 1.1,

$$20\log|G(f_\pi)| = 60dB \tag{1.4.4}$$
$$\implies 20\log|G(f_\pi)| - 20\log\left(\frac{1}{H(f_\pi)}\right) = (60-85)\,dB \tag{1.4.5}$$
$$\implies GM = \left|20\log|G(f_\pi)H(f_\pi)|\right| = 25dB \tag{1.4.6}$$

1.5. Break the Transfer Function $G(f)$ into Simple Blocks and Create a Block Diagram for $G(f)$.
**Solution:**

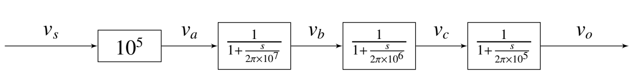


Fig. 1.5

1.6. Find the Gain of RC-Circuit shown below 1.6 and also identify the location of Poles.

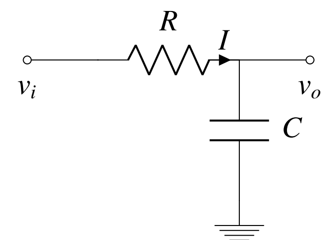


Fig. 1.6

**Solution:**

$$I = \frac{v_i}{R+\frac{1}{Cs}} \tag{1.6.1}$$
$$v_o = I\times\frac{1}{Cs} \tag{1.6.2}$$
$$v_o = \frac{v_i\times\frac{1}{Cs}}{R+\frac{1}{Cs}} \tag{1.6.3}$$
$$\frac{v_o}{v_i} = \frac{1}{RCs+1} \tag{1.6.4}$$
$$Gain = \frac{v_o}{v_i} = \frac{1}{RCs+1} \tag{1.6.5}$$

So, there is a Pole at frequency $f = \frac{1}{2\pi RC}$ for the Transfer Function of Gain.

1.7. Find the Gain of Operational Amplifier. The circuit diagram of Equivalent Circuit is 1.7.

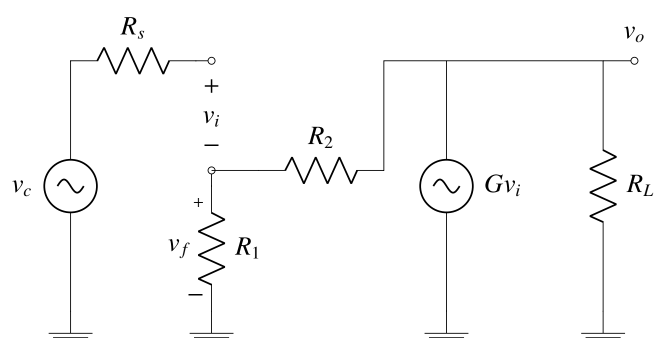


Fig. 1.7

**Solution:**
Applying KVL and KCL,

$$v_o = Gv_i \tag{1.7.1}$$

As no current flows through $R_s$,

$$v_i = v_c - v_f \tag{1.7.2}$$

$$v_f = \frac{R_1}{R_1 + R_2} v_o \tag{1.7.3}$$

$$v_i = \frac{v_o}{G} \tag{1.7.4}$$

$$\frac{v_o}{G} = v_c - \frac{R_1}{R_1 + R_2} v_o \tag{1.7.5}$$

$$\frac{v_o}{v_c} = \frac{G}{1 + G\frac{R_1}{R_1+R_2}} \tag{1.7.6}$$

So, Gain of the Circuit is $\frac{G}{1+G\frac{R_1}{R_1+R_2}}$

1.8. Design a Circuit Model that follows the Transfer Function $G(s)$

**Solution:**

Our Design for Modelling the Transfer Function is based on Poles of RC-Circuit and Gain of Operational Amplifier.

So, the Circuit Diagram is,

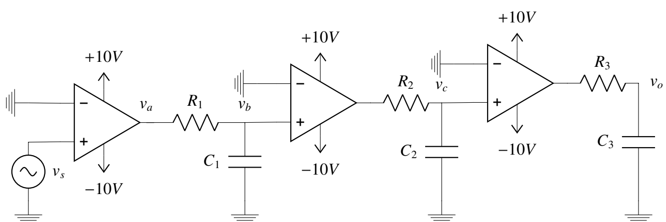


Fig. 1.8

Assuming, Open-Loop Gain of Operational Amplifier is $10^5$ and also assuming Operational Amplifier doesnt have any Poles.
Equivalent Circuit of the circuit is

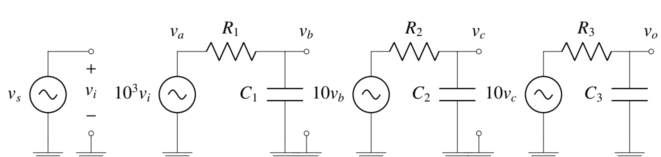


Fig. 1.8

The cascade of RC Circuits are used to introduce poles in the circuit and Op-Amp are used to achieve the Gain required.

At the First Operational Amplifier,

$$v_i = v_s \tag{1.8.1}$$

$$v_a = 10^3 v_i \tag{1.8.2}$$

$$v_a = 10^3 v_s \tag{1.8.3}$$

At the first RC-Circuit,

$$2\pi RC = 10^{-7} \tag{1.8.4}$$

$$v_b = \frac{v_a}{1 + j\frac{f}{10^7}} \tag{1.8.5}$$

$$v_b = \frac{10^3 v_i}{1 + j\frac{f}{10^7}} \tag{1.8.6}$$

At the Second Operational Amplifier and Second RC-Circuit,

$$2\pi RC = 10^{-6} \tag{1.8.7}$$

$$v_c = \frac{10 v_b}{1 + j\frac{f}{10^6}} \tag{1.8.8}$$

$$v_c = \frac{10^4 v_i}{(1 + j\frac{f}{10^6})(1 + j\frac{f}{10^7})} \tag{1.8.9}$$

At the Third Operational Amplifier and Third RC-Circuit,

$$2\pi RC = 10^{-5} \tag{1.8.10}$$

$$v_o = \frac{10 v_c}{1 + j\frac{f}{10^5}} \tag{1.8.11}$$

$$v_o = \frac{10^5 v_i}{(1 + j\frac{f}{10^5})(1 + j\frac{f}{10^6})(1 + j\frac{f}{10^7})} \tag{1.8.12}$$

The RC Circuits introduces poles at $f = 10^7 Hz, 10^6 Hz, 10^5 Hz$ respectively from left to right. The Op-Amps introduce a gain $10^3, 10, 10$. THe second and third Op-Amps act as a buffer. So, the value of $v_o$ is

$$v_o = \frac{10^5 v_i}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \tag{1.8.13}$$

So, Open-Loop Gain is

$$G = \frac{10^5}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \tag{1.8.14}$$

1.9. Design a Circuit Model that follows the Feedback Transfer Function $H(s)$

**Solution:**

On Bode Plot is $H$ is independent of frequency. So, $H$ should not involve any Reactive Elements. So, $H$ is a combination of Resistors or a Voltage Divider.

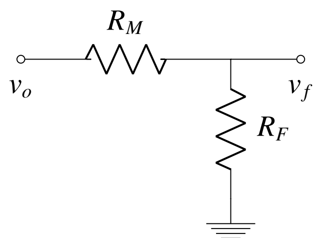


Fig. 1.9

$$v_f = \frac{R_F}{R_F + R_M} \times v_o \tag{1.9.1}$$

Assuming,

| Circuit Element | Value |
|---|---|
| $R_M$ | $1.778 \times 10^5 \Omega$ |
| $R_F$ | $10\Omega$ |

TABLE 1.9

On substituting,

$$v_f = \frac{10}{10 + 1.778 \times 10^5} \times v_o \tag{1.9.2}$$

$$v_f \approx 5.623 \times 10^{-5} v_o \tag{1.9.3}$$

$$\frac{v_f}{v_o} \approx 5.623 \times 10^{-5} \tag{1.9.4}$$

$$H(s) = 5.623 \times 10^{-5} \tag{1.9.5}$$

1.10. Draw the Magnitude and Phase Bode Plots of $G(f)$
**Solution:** Bode Plot is 1.10
Python Code for Bode Plot is at

```
codes/ee18btech11014/Bode_Plot.py
```

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1.11. Design a Closed-Loop Transfer Function by combining both the Open-Loop and Feedback Circuits. Also draw its Equivalent Circuit
**Solution:**
The Closed-Loop Circuit is
The Equivalent Circuit of Closed-Loop Circuit is

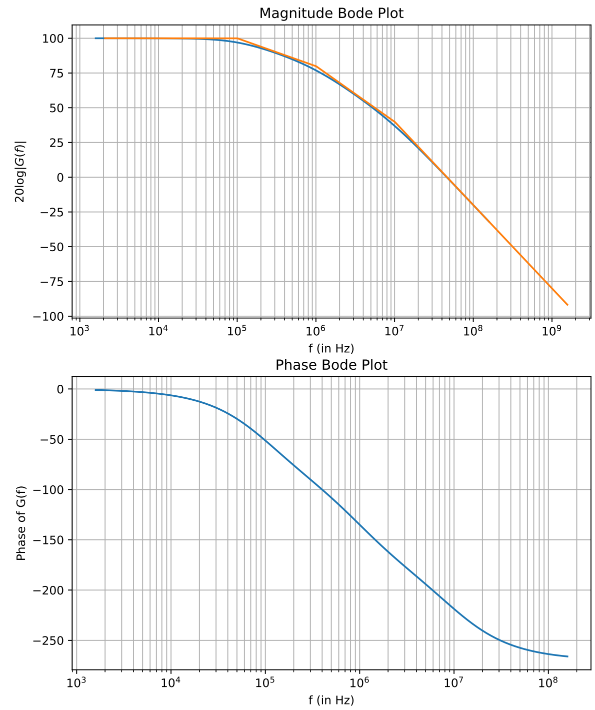


Fig. 1.10: Magnitude Bode Plot

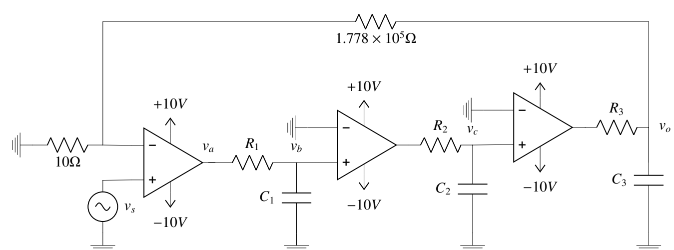


Fig. 1.11

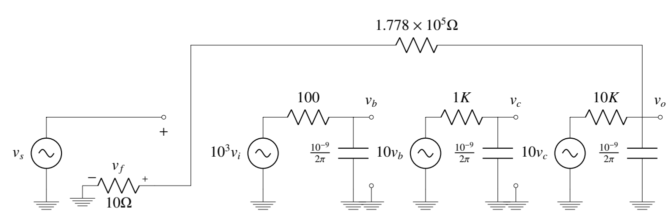


Fig. 1.11

From the Equivalent Circuit Diagram,

$$G(s) = \frac{10^5}{\left(1 + \frac{s}{2\pi 10^5}\right)\left(1 + \frac{s}{2\pi 10^6}\right)\left(1 + \frac{s}{2\pi 10^7}\right)} \tag{1.11.1}$$

$$H(s) = \frac{v_f}{v_o} = 5.623 \times 10^{-5} \tag{1.11.2}$$

The Closed-Loop Gain,

$$v_i = v_s - v_f \tag{1.11.3}$$

$$\frac{v_o}{G} = v_s - Hv_o \tag{1.11.4}$$

$$\frac{v_o}{v_s} = \frac{G}{1 + GH} \tag{1.11.5}$$

So, the Closed-Loop Gain,

$$T(s) = \frac{v_o}{v_s} \tag{1.11.6}$$

$$T(s) = \frac{10^5}{\left(1 + \frac{s}{2\pi 10^5}\right)\left(1 + \frac{s}{2\pi 10^6}\right)\left(1 + \frac{s}{2\pi 10^7}\right) + 5.623} \tag{1.11.7}$$

## 2 Assignment

2.1. Find the frequencies for which phase margins are 90° and 45° respectively?
**Solution:** $\because \angle H(f) = 1$, Let Phase Margin be $\alpha = 90°$. Then,

$$\angle G(f_{90}) H(f_{90}) = \angle G(f_{90}) = 90° - 180° \tag{2.1.1}$$

$$= -90° \tag{2.1.2}$$

$$\implies |G(f_{90}) H(f_{90})| = 1 \tag{2.1.3}$$

So, by the definition of Phase-Margin, at $\phi = -90°$ , $|GH| = 1$. The value of $\phi = -90°$ between poles $f = 10^5 Hz, 10^6 Hz$. Assuming the Poles are farther apart,

$$\tan^{-1}\left(\frac{f}{10^7}\right) \approx 0 \tag{2.1.4}$$

where $10^5 < f < 10^6$
So,

$$-\tan^{-1}\left(f/10^5\right) - \tan^{-1}\left(f/10^6\right) = -90 \tag{2.1.5}$$

$$\tan^{-1}\left(f/10^5\right) + \tan^{-1}\left(f/10^6\right) = 90 \tag{2.1.6}$$

$$\tan^{-1}\left(f/10^5\right) = 90 - \tan^{-1}\left(f/10^6\right) \tag{2.1.7}$$

$$\tan^{-1}\left(f/10^5\right) = \cot^{-1}\left(f/10^6\right) \tag{2.1.8}$$

$$\tan^{-1}\left(f/10^5\right) = \tan^{-1}\left(10^6/f\right) \tag{2.1.9}$$

$$f^2 = 10^{11} \tag{2.1.10}$$

$$f = 3.162 \times 10^5 \tag{2.1.11}$$

So, the approximate value of $f$ at which Phase Margin is 90° is $f = 3.162 \times 10^5 Hz$.

Similarly let Phase Margin be $\alpha = 45°$. Then,

$$\alpha = \phi - (-180°) \tag{2.1.12}$$

$$\phi = -180° + \alpha \tag{2.1.13}$$

$$\phi = -135° \tag{2.1.14}$$

So, by the definition of Phase-Margin, at $\phi = -135°$ , $|GH| = 1$. The value of $\phi = -135°$ aproximately at poles $f = 10^6 Hz$.
So, the approximate value of $f$ at which Phase Margin is 45° is $f = 10^6$.

2.2. Find the minimum values of Closed-Loop Voltage Gain for which phase margins are 90° and 45° respectively
**Solution:**
For $\alpha = 90°$,

$$f = 3.162 \times 10^5 \tag{2.2.1}$$

By substituting $f$ in Open-Loop Gain $G(f)$ (assuming poles are far part),

$$G(f) = 200 - 20log(3.162 \times 10^5) \tag{2.2.2}$$

$$G(f) = 90dB \tag{2.2.3}$$

$$G = 3.1625 \times 10^4 \tag{2.2.4}$$

At that $f = 3.162 \times 10^5$,

$$H = \frac{1}{G} \tag{2.2.5}$$

$$H = 3.162 \times 10^{-5} \tag{2.2.6}$$

The minimum value of Closed-Loop Gain occurs at $|GH| \gg 1$ and the value of Closed-Loop Gain is $T = \frac{1}{H}$

$$T = \frac{1}{H} = 3.1625 \times 10^4 \tag{2.2.7}$$

**So, The minimum value of Closed-Loop Gain with Phase Margin equal to $\alpha = 90°$ is $T_{min} = 3.1625 \times 10^4$.**

For $\alpha = 45°$,

$$f = 10^6 \tag{2.2.8}$$

By substituting $f$ in Open-Loop Gain $G(f)$

(assuming poles are far part),

$$G(f) = 200 - 20log(10^6) \quad (2.2.9)$$
$$G(f) = 80dB \quad (2.2.10)$$
$$G = 10^4 \quad (2.2.11)$$

At that $f = 10^6$,

$$H = \frac{1}{G} \quad (2.2.12)$$
$$H = 10^{-4} \quad (2.2.13)$$

The minimum value of Closed-Loop Gain occurs at $|GH| \gg 1$ and the value of Closed-Loop Gain is $T = \frac{1}{H}$

$$T = \frac{1}{H} = 10^4 \quad (2.2.14)$$

**So, The minimum value of Closed-Loop Gain with Phase Margin equal to $\alpha = 45°$ is $T_{min} = 10^4$.**

2.3. Design a Feedback circuit for Phase Margin $\alpha = 45°$.
**Solution:**

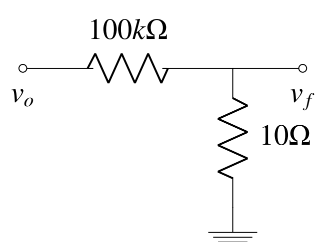


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Fig. 2.3

$$v_f = \frac{10}{10 + 10^5} \times v_o \quad (2.3.1)$$
$$v_f \approx 10^{-4} v_o \quad (2.3.2)$$
$$\frac{v_f}{v_o} \approx 10^{-4} \quad (2.3.3)$$
$$H(s) = 10^{-4} \quad (2.3.4)$$

2.4. Design a Feedback circuit for Phase Margin $\alpha = 90°$.
**Solution:**

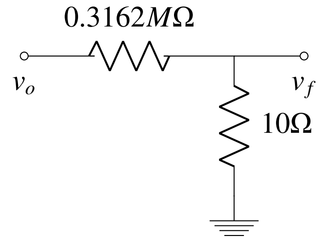


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Fig. 2.4

$$v_f = \frac{10}{10 + 3.162 \times 10^5} \times v_o \quad (2.4.1)$$
$$v_f \approx 3.162 \times 10^{-5} v_o \quad (2.4.2)$$
$$\frac{v_f}{v_o} \approx 3.162 \times 10^{-5} \quad (2.4.3)$$
$$H(s) = 3.162 \times 10^{-5} \quad (2.4.4)$$

2.5. Design a Closed-Loop Transfer Function by combining both the Open-Loop and Feedback Circuits for phase Margin $\alpha = 45°$. Also draw its Equivalent Circuit
**Solution:**
The Closed-Loop Circuit is

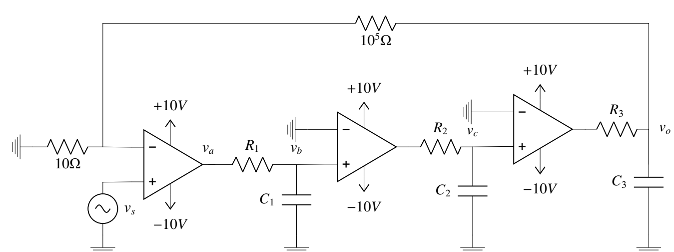


Fig. 2.5

The Equivalent Circuit of Closed-Loop Circuit is

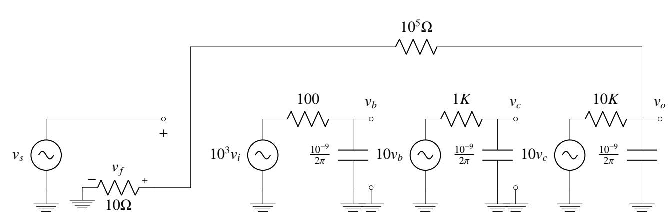


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Fig. 2.5

From the Equivalent Circuit Diagram,

$$G(s) = \frac{10^5}{\left(1 + \frac{s}{2\pi 10^5}\right)\left(1 + \frac{s}{2\pi 10^6}\right)\left(1 + \frac{s}{2\pi 10^7}\right)} \tag{2.5.1}$$

$$H(s) = \frac{v_f}{v_o} = 10^{-4} \tag{2.5.2}$$

The Closed-Loop Gain,

$$v_i = v_s - v_f \tag{2.5.3}$$

$$\frac{v_o}{G} = v_s - Hv_o \tag{2.5.4}$$

$$\frac{v_o}{v_s} = \frac{G}{1+GH} \tag{2.5.5}$$

So, the Closed-Loop Gain,

$$T(s) = \frac{v_o}{v_s} = \frac{10^5}{10 + \left(1 + s\frac{s}{2\pi 10^5}\right)\left(1 + \frac{s}{2\pi 10^6}\right)\left(1 + j\frac{s}{2\pi 10^7}\right)} \tag{2.5.6}$$

2.6. Design a Closed-Loop Transfer Function by combining both the Open-Loop and Feedback Circuits for phase Margin $\alpha = 90°$. Also draw its Equivalent Circuit

**Solution:**

The Closed-Loop Circuit is

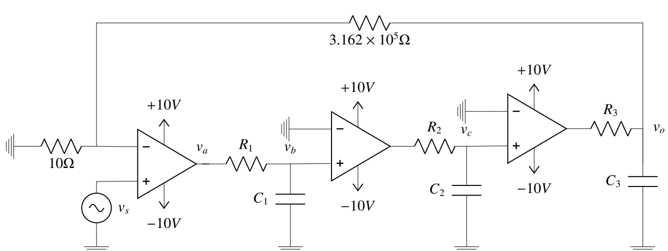


Fig. 2.6

The Equivalent Circuit of Closed-Loop Circuit is

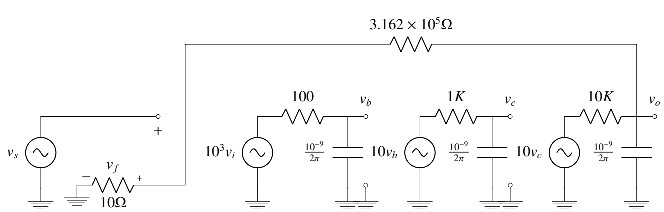


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Fig. 2.6

From the Equivalent Circuit Diagram,

$$G(s) = \frac{10^5}{\left(1 + \frac{s}{2\pi 10^5}\right)\left(1 + \frac{s}{2\pi 10^6}\right)\left(1 + \frac{s}{2\pi 10^7}\right)} \tag{2.6.1}$$

$$H(s) = \frac{v_f}{v_o} = 3.162 \times 10^{-5} \tag{2.6.2}$$

The Closed-Loop Gain,

$$v_i = v_s - v_f \tag{2.6.3}$$

$$\frac{v_o}{G} = v_s - Hv_o \tag{2.6.4}$$

$$\frac{v_o}{v_s} = \frac{G}{1+GH} \tag{2.6.5}$$

So, the Closed-Loop Gain,

$$T(s) = \frac{v_o}{v_s} = \frac{10^5}{3.162 + \left(1 + s\frac{s}{2\pi 10^5}\right)\left(1 + \frac{s}{2\pi 10^6}\right)\left(1 + j\frac{s}{2\pi 10^7}\right)} \tag{2.6.6}$$