EE2015: Assignment 1 - Fourier Analysis

Deadline: Wed, 7th Aug, 2:00 PM

Implementation and submission guidelines: You are required to perform the necessary coding for this assignment in Python. You may use *numpy*, *matplotlib* and other libraries. You will need to submit a single pdf file containing your source code and remarks interpreting your results.

1. **Synthesis of signals**

   We know from the Fourier theory that any periodic signal can be expressed as a sum of harmonic functions. Consider t in the range $0 \leq t \leq 4\pi$ and answer the following:

   (a) Plot f_1 given below for $N = 1, 5, 20$

   $$f_1(t) = \sum_{n=1,3,5..}^{N} \frac{\sin(nt)}{n}$$

   (b) Plot f_2 given below for $N = 1, 5, 20$

   $$f_2(t) = \sum_{n=1,2,3..}^{N} \frac{\sin(nt)}{n}$$

   (c) You would observe that larger N gives a better approximation of square and ramp signals for part (a) and (b) respectively. Plot the Fourier spectrum (Amplitude vs Frequency components) of f_1 and f_2 respectively.

   (d) How would you modify the above summation if you are required to obtain an even function (*e.g.* Triangle function)?

   (e) You may have also observed that f_1 and f_2 have duty cycle of 50%. Can you modify the summation such that duty cycle can adjusted to a desired duty cycle D, and amplitude A?

2. **Fourier series for square and ramp signals**

   (a) Consider two periodic signals $f_1(t)$ and $f_2(t)$ of period 1 that are defined as

   $$f_1(t) = \begin{cases} +1 & 0 \leq t < \frac{1}{2} \\ -1 & \frac{1}{2} \leq t < 1 \end{cases} \tag{1}$$

   $$f_2(t) = \begin{cases} \frac{1}{2} + t & -\frac{1}{2} \leq t < 0 \\ \frac{1}{2} - t & 0 \leq t < \frac{1}{2} \end{cases} \tag{2}$$

      (a) Plot the signals in the time period $-2 < t < 2$.

      (b) Determine if the signals

         (a) are continuous or discontinuous,

         (b) are Odd or even,

         (c) posses half wave symmetry and

         (d) posses quarter wave symmetry.

      (e) Assume that the signals are to be represented in Fourier series using the trigonometric form as

      $$f(t) = \sum_{k=0}^{N} (a_k \cos(2\pi kt) + b_k \sin(2\pi kt)) \tag{3}$$

      Compute the Fourier coefficients a_k and b_k for both the signals.

(f) Assume that the signals are to be represented in Fourier series using the exponential form as

$$f(t) \;=\; \sum_{k=-n}^{k=n} C_k e^{i(2\pi kt)} \tag{4}$$

Compute the Fourier coefficients c_k for both the signals.

(g) Does the Fourier series converge for $f_1(t)$ and $f_2(t)$. State the relevant reasons to justify the answer.

(h) Plot the Fourier spectrum (Amplitude vs Freqency components) of $f_1(t)$ and $f_2(t)$ respectively for $N \leq 15$.

(i) Using first principles such as continuity/discontinuity, odd/even of a function, state the reasons why the Fourier series of $f_1(t)$ and $f_2(t)$ have higher order Fourier coefficients.

3. **Sampling and windowing** For this problem, recall the DFT of an $N-$length signal $x[0], x[1], \ldots x[N-1]$ is given by

$$\mathcal{F}x[m] = \sum_{n=0}^{N-1} x[n]e^{-2\pi imn/N},$$

and can be calculated using `numpy.fft`. Consider the signal defined by $f(t) = e^{-a^2\pi t^2}$. See Fig 1 for an illustration. The (continuous) Fourier transform $\mathcal{F}f(s)$ of $f(t)$ is defined as

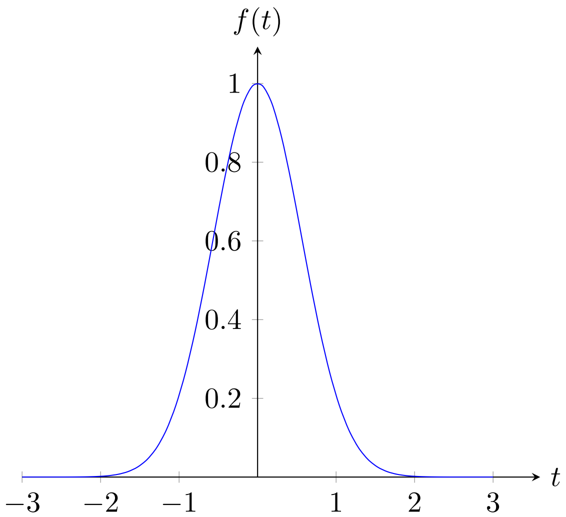

Figure 1: Problem 3

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ist}dt, \tag{5}$$

where s represents frequency. It can be shown that

$$\mathcal{F}f(s) = \frac{1}{a}e^{-\pi s^2/a^2}.$$

(a) Sketch a plot of $\mathcal{F}f(s)$. Explain what you observe as you vary a.

For the rest of the problem we will attempt to compute the Fourier transform of the time-domain signal $f(t)$ using numpy.fft.

(b) Assume $a = 1$, and pick a sampling rate R_{time} to construct the sequence $\{f(nR_{\text{time}})\}$ for $n = -\infty$ to ∞. Write an approximation the integral from (5) using the samples $\{f(nR_{\text{time}})\}$. You should get an infinite sum.

(c) To make the sum finite, we consider the samples $f(nR_{\text{time}})$ to be non zero only when

$$-L/2 \leq nR_{\text{time}} < L/2.$$

This effectively multiplies the time domain function $f(t)$ with a rectangular *windowing function* $w_1(t)$ as shown in Fig 2. Modify the approximation to (5) that you constructed

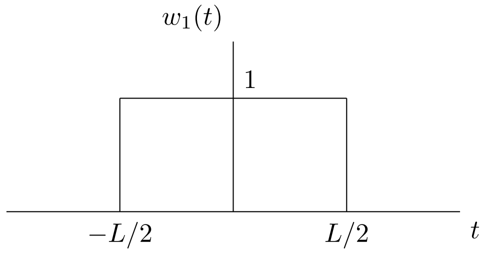

Figure 2: Rectangular window

in (b) above using the window function.

(d) The expression you get in (b) gives a continuous function of the frequency s. We evaluate it at $s = mR_{\text{freq}}$ for m such that

$$\frac{-B}{2} \leq mR_{\text{freq}} < \frac{B}{2}.$$

Write an (approximate) expression for $\mathcal{F}f(mR_{\text{freq}})$.

(e) The system parameters so far are L, B, R_{time} and R_{freq}. Suppose we set $LR_{\text{freq}} = BR_{\text{time}} = 1$. In terms of these system parameters, how many samples need to be stored in the time domain ? How many values are computed in the frequency domain ? Explain why we would like to pick $LR_{\text{freq}} = BR_{\text{time}} = 1$.

(f) Use `numpy.fft` to evaluate the approximation to $\mathcal{F}f(s)$ that you derived in (d).

(g) Repeat (e) by taking different values of the system parameters, and plot the resulting approximations to $\mathcal{F}f(mR_{\text{freq}})$. Explain your observations.

(h) Consider the following alternate choices of windowing functions given by

$$w_2(t) = \begin{cases} 1 - \frac{2|t|}{L} \text{ for } |t| \leq L/2 \\ 0 \text{ else.} \end{cases} \quad , \quad w_3(t) = \begin{cases} \sin^2 \frac{2\pi t}{L} \text{ for } -L/2 \leq t \leq L/2 \\ 0 \text{ else.} \end{cases}$$

Plot the approximation to $\mathcal{F}f(s)$ for each of these windows and compare.

(i) Repeat the analysis for varying system parameters and window functions for the signal

$$g(t) = \cos 2\pi t + 0.5 \sin 4\pi t,$$

and explain your observations.

4. **And yet it flies**[1]

Watch the video 'chopper' available at

`http://www.youtube.com/watch?v=bZCUB_BiY_4`

[1] Osgood BG. Lectures on the Fourier Transform and Its Applications. American Mathematical Soc.; 2019

Suppose the frame rate of the video camera is R_1, i.e., the camera is taking R_1 still shots per second; and the rotation rate of the main rotor is R_2 rotations per second.

(a) Explain what you observe in the video.

(b) Suppose R_1 is fixed and the chopper has 5 rotor blades. What values of R_2 (expressed in terms of R_1) cause the rotor to appear stationary as in the video?

(c) In part (a), we assumed that the chopper has 5 rotor blades. Is this assumption valid? If you had seen 6 blades in the video, how many blades do you think the chopper has? Explain.