Control Systems

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Contents

***Abstract*—This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/codes
```

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1 Signal Flow Graph

2 Gain of Feedback Circuits

2.1 Estimation of Voltage Gain

2.1.1. Consider the Magnitude Bode Plot and Phase Bode Plot 2.1.1 of Open-Loop Transfer Function of an Amplifier. Estimate the Open-Loop Transfer Function. (Assume $'A'$ as $'G'$ and $'\beta'$ as $'H'$)

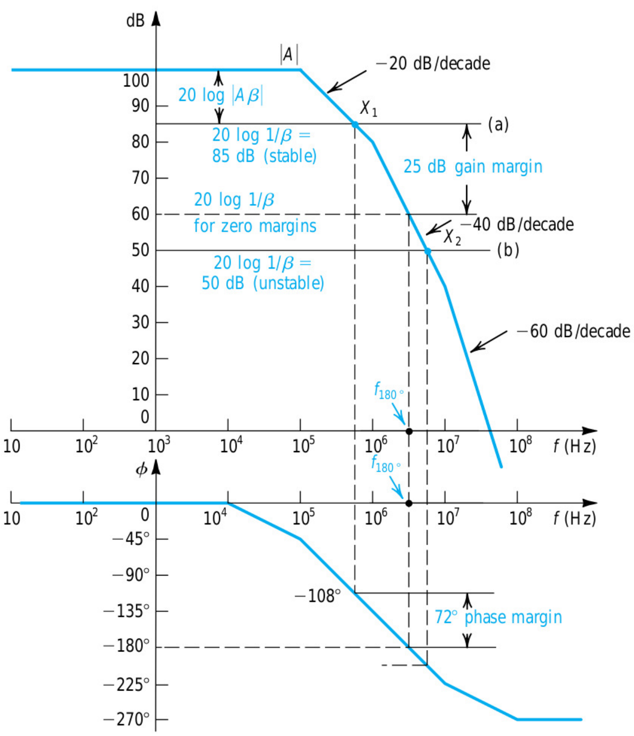

Fig. 2.1.1: Magnitude and Phase Bode Plot

Solution: Let $G(f)$ be the Open-Loop Transfer Function,

$$G(f) = \begin{cases} 100 & 0 < f < 10^5 \\ 200 - 20log(f) & 10^5 < f < 10^6 \\ 320 - 40log(f) & 10^6 < f < 10^7 \\ 460 - 60log(f) & 10^7 < f \end{cases} \tag{2.1.1.1}$$

$$\nabla G(f) = \frac{d(G(f))}{d(log(f))} = \begin{cases} 0 & 0 < f < 10^5 \\ -20 & 10^5 < f < 10^6 \\ -40 & 10^6 < f < 10^7 \\ -60 & 10^7 < f \end{cases} \tag{2.1.1.2}$$

As we know that, **When a pole is encountered the slope always decreases by 20 dB/decade** and **When a zero is encountered the slope always increases by 20 dB/decade**. So, by observing 2.1.1 it can be concluded that we are having Poles at $f = 10^5 Hz, 10^6 Hz, 10^7 Hz$ and No Zeros.

So, the Open-Loop Transfer Function $G(f)$ is

$$G(f) = \frac{10^5}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \tag{2.1.1.3}$$

2.1.2. Calculate the Phase of Open-Loop Transfer Function.
Solution:
Phase of Open-Loop Transfer Function = ϕ

$$\phi = -\left[\tan^{-1}\left(f/10^5\right) + \tan^{-1}\left(f/10^6\right) + \tan^{-1}\left(f/10^7\right)\right] \tag{2.1.2.1}$$

2.1.3. Determine the Closed-Loop Voltage Gain of the System assuming $|GH| \gg 1$ and also assuming the block diagram of Control System is 2.1.3

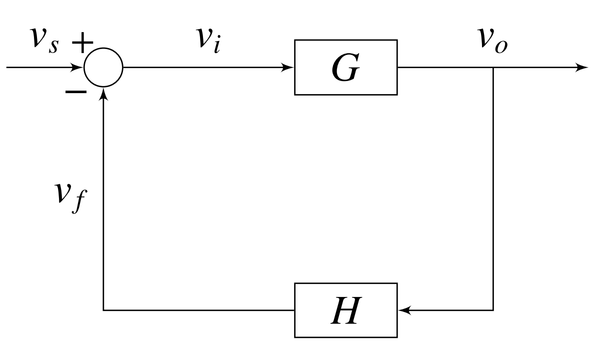

Fig. 2.1.3

Solution:
The Closed-Loop Voltage Gain of the Control System is

$$T = \frac{V_o}{V_s} = \frac{G}{1 + GH} \tag{2.1.3.1}$$

$$20\log(T) = 20\log(G) - 20\log(1 + GH) \tag{2.1.3.2}$$

Considering the assumption $|GH| \gg 1$, It can be written as

$$20\log(1 + GH) = 20\log(GH) \tag{2.1.3.3}$$

So,

$$20\log(T) = 20\log(G) - 20\log(GH) \tag{2.1.3.4}$$

$$20\log(T) = -20\log(H) \tag{2.1.3.5}$$

$$20\log(T) = 20\log(\frac{1}{H}) \tag{2.1.3.6}$$

$$T = \frac{1}{H} \tag{2.1.3.7}$$

So, The value of Closed-Loop Voltage Gain of the Control System under the assumption, $|GH| \gg 1$ is $T = \frac{1}{H}$

2.1.4. What is the value of Loop-Gain?
Solution:
The value of Loop-Gain can be calculated by the difference of 2-curves $20\log|A|$ and $20log(\frac{1}{H})$. The difference between the two curves will be

$$20\log|G| - 20\log\frac{1}{H} = 20\log|GH| \tag{2.1.4.1}$$

2.1.5. Define Phase-Margin
Solution:
Phase-Margin: The phase margin is defined as the angle in degrees by which the phase angle is smaller than $-180°$ at the gain crossover, the gain crossover being the frequency at which the open-loop gain first reaches 1.

2.1.6. Find the frequencies for which phase margins are 90° and 45° respectively?
Solution:
Let Phase Margin be $\alpha = 90°$. Then,

$$\alpha = \phi - (-180°) \tag{2.1.6.1}$$

$$\phi = -180° + \alpha \tag{2.1.6.2}$$

$$\phi = -90° \tag{2.1.6.3}$$

So, by the definition of Phase-Margin, at $\phi = -90^\circ$, $|GH| = 1$. The value of $\phi = -90^\circ$ between poles $f = 10^5 Hz, 10^6 Hz$. Assuming the Poles are farther apart,

$$\tan^{-1}(\frac{f}{10^7}) \approx 0 \tag{2.1.6.4}$$

where $10^5 < f < 10^6$
So,

$$-\tan^{-1}\left(f/10^5\right) - \tan^{-1}\left(f/10^6\right) = -90 \tag{2.1.6.5}$$

$$\tan^{-1}\left(f/10^5\right) + \tan^{-1}\left(f/10^6\right) = 90 \tag{2.1.6.6}$$

$$\tan^{-1}\left(f/10^5\right) = 90 - \tan^{-1}\left(f/10^6\right) \tag{2.1.6.7}$$

$$\tan^{-1}\left(f/10^5\right) = \cot^{-1}\left(f/10^6\right) \tag{2.1.6.8}$$

$$\tan^{-1}\left(f/10^5\right) = \tan^{-1}\left(10^6/f\right) \tag{2.1.6.9}$$

$$f^2 = 10^{11} \tag{2.1.6.10}$$

$$f = 3.162 \times 10^5 \tag{2.1.6.11}$$

So, the approximate value of f at which Phase Margin is 90° is $f = 3.162 \times 10^5 Hz$.

Similarly let Phase Margin be $\alpha = 45^\circ$. Then,

$$\alpha = \phi - (-180^\circ) \tag{2.1.6.12}$$

$$\phi = -180^\circ + \alpha \tag{2.1.6.13}$$

$$\phi = -135^\circ \tag{2.1.6.14}$$

So, by the definition of Phase-Margin, at $\phi = -135^\circ$, $|GH| = 1$. The value of $\phi = -135^\circ$ aproximately at poles $f = 10^6 Hz$.
So, the approximate value of f at which Phase Margin is 45° is $f = 10^6$.

2.1.7. Find the minimum values of Closed-Loop Voltage Gain for which phase margins are 90° and 45° respectively
Solution:
For $\alpha = 90^\circ$,

$$f = 3.162 \times 10^5 \tag{2.1.7.1}$$

By substituting f in Open-Loop Gain $G(f)$ (assuming poles are far part),

$$G(f) = 200 - 20log(3.162 \times 10^5) \tag{2.1.7.2}$$

$$G(f) = 90dB \tag{2.1.7.3}$$

$$G = 3.1625 \times 10^4 \tag{2.1.7.4}$$

At that $f = 3.162 \times 10^5$,

$$H = \frac{1}{G} \tag{2.1.7.5}$$

$$H = 3.162 \times 10^{-5} \tag{2.1.7.6}$$

The minimum value of Closed-Loop Gain occurs at $|GH| \gg 1$ and the value of Closed-Loop Gain is $T = \frac{1}{H}$

$$T = \frac{1}{H} = 3.1625 \times 10^4 \tag{2.1.7.7}$$

So, The minimum value of Closed-Loop Gain with Phase Margin equal to $\alpha = 90^\circ$ is $T_{min} = 3.1625 \times 10^4$.

For $\alpha = 45^\circ$,

$$f = 10^6 \tag{2.1.7.8}$$

By substituting f in Open-Loop Gain $G(f)$ (assuming poles are far part),

$$G(f) = 200 - 20log(10^6) \tag{2.1.7.9}$$

$$G(f) = 80dB \tag{2.1.7.10}$$

$$G = 10^4 \tag{2.1.7.11}$$

At that $f = 10^6$,

$$H = \frac{1}{G} \tag{2.1.7.12}$$

$$H = 10^{-4} \tag{2.1.7.13}$$

The minimum value of Closed-Loop Gain occurs at $|GH| \gg 1$ and the value of Closed-Loop Gain is $T = \frac{1}{H}$

$$T = \frac{1}{H} = 10^4 \tag{2.1.7.14}$$

So, The minimum value of Closed-Loop Gain with Phase Margin equal to $\alpha = 45^\circ$ is $T_{min} = 10^4$.

2.1.8. Break the Transfer Function $G(f)$ into Simple Blocks and Create a Block Diagram for $G(f)$.
Solution:

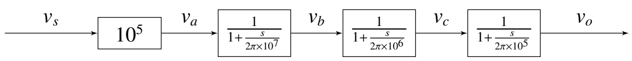

Fig. 2.1.8

2.1.9. Find the Gain of RC-Circuit shown below 2.1.9 and also identify the location of Poles.

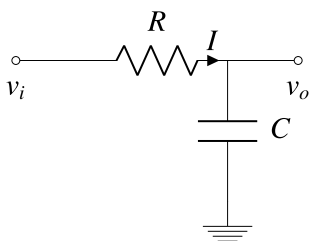

Fig. 2.1.9

Solution:

$$I = \frac{v_{input}}{R + \frac{1}{Cs}} \tag{2.1.9.1}$$

$$v_{output} = I \times \frac{1}{Cs} \tag{2.1.9.2}$$

$$v_{output} = \frac{v_{input} \times \frac{1}{Cs}}{R + \frac{1}{Cs}} \tag{2.1.9.3}$$

$$\frac{v_{output}}{v_{input}} = \frac{1}{RCs + 1} \tag{2.1.9.4}$$

$$s = j2\pi f \tag{2.1.9.5}$$

$$Gain = \frac{v_{output}}{v_{input}} = \frac{1}{j2\pi RCf + 1} \tag{2.1.9.6}$$

So, there is a Pole at frequency $f = \frac{1}{2\pi RC}$ for the Transfer Function of Gain.

2.1.10. Find the Gain of Operational Amplifier. The circuit diagram of Equivalent Circuit is 2.1.10.

Solution:
Applying KVL and KCL,

$$v_o = Gv_i \tag{2.1.10.1}$$

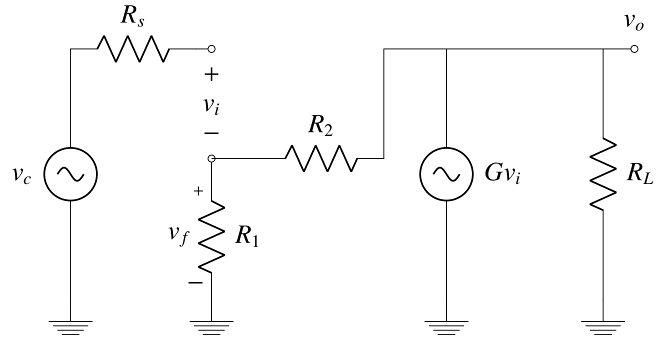

Fig. 2.1.10

As no current flows through R_s,

$$v_i = v_c - v_f \tag{2.1.10.2}$$

$$v_f = \frac{R_1}{R_1 + R_2} v_o \tag{2.1.10.3}$$

$$v_i = \frac{v_o}{G} \tag{2.1.10.4}$$

$$\frac{v_o}{G} = v_c - \frac{R_1}{R_1 + R_2} v_o \tag{2.1.10.5}$$

$$\frac{v_o}{v_c} = \frac{G}{1 + G\frac{R_1}{R_1+R_2}} \tag{2.1.10.6}$$

So, Gain of the Circuit is $\frac{G}{1+G\frac{R_1}{R_1+R_2}}$

2.1.11. Design a Circuit Model that follows the Transfer Function $G(f)$
Solution:
Our Design for Modelling the Transfer Function is based on Poles of RC-Circuit and Gain of Operational Amplifier.

So, the Circuit Diagram is,

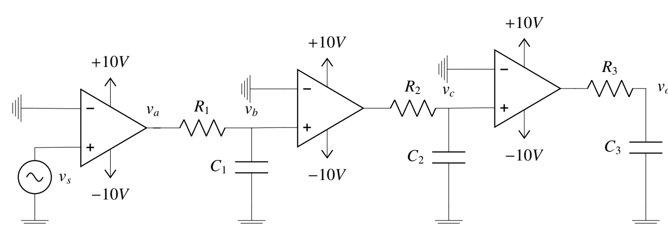

Fig. 2.1.11

Assuming, Open-Loop Gain of Operational Amplifier is 10^5 and also assuming Operational Amplifier doesnt have any Poles.
Equivalent Circuit of the circuit is
The cascade of RC Circuits are used to introduce poles in the circuit and Op-Amp are

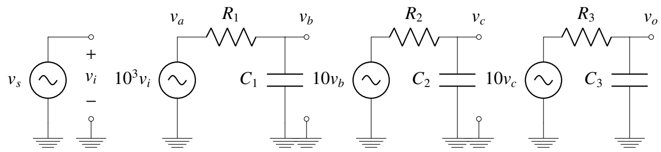

Fig. 2.1.11

used to achieve the Gain required.

At the First Operational Amplifier,

$$v_i = v_s \tag{2.1.11.1}$$
$$v_a = 10^3 v_i \tag{2.1.11.2}$$
$$v_a = 10^3 v_s \tag{2.1.11.3}$$

At the first RC-Circuit,

$$2\pi RC = 10^{-7} \tag{2.1.11.4}$$
$$v_b = \frac{v_a}{1 + j\frac{f}{10^7}} \tag{2.1.11.5}$$
$$v_b = \frac{10^3 v_i}{1 + j\frac{f}{10^7}} \tag{2.1.11.6}$$

At the Second Operational Amplifier and Second RC-Circuit,

$$2\pi RC = 10^{-6} \tag{2.1.11.7}$$
$$v_c = \frac{10 v_b}{1 + j\frac{f}{10^6}} \tag{2.1.11.8}$$
$$v_c = \frac{10^4 v_i}{(1 + j\frac{f}{10^6})(1 + j\frac{f}{10^7})} \tag{2.1.11.9}$$

At the Third Operational Amplifier and Third RC-Circuit,

$$2\pi RC = 10^{-5} \tag{2.1.11.10}$$
$$v_o = \frac{10 v_c}{1 + j\frac{f}{10^5}} \tag{2.1.11.11}$$

$$v_o = \frac{10^5 v_i}{(1 + j\frac{f}{10^5})(1 + j\frac{f}{10^6})(1 + j\frac{f}{10^7})} \tag{2.1.11.12}$$

The RC Circuits introduces poles at $f = 10^7 Hz, 10^6 Hz, 10^5 Hz$ respectively from left to right. The Op-Amps introduce a gain $10^3, 10, 10$. THe second and third Op-Amps act as a buffer. So, the value of v_o is

$$v_o = \frac{10^5 v_i}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \tag{2.1.11.13}$$

So, Open-Loop Gain is

$$G = \frac{10^5}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \tag{2.1.11.14}$$

2.1.12. Design a Circuit Model that follows the Feedback Transfer Function $H(f)$
Solution:
On Bode Plot is H is independent of frequency. So, H should not involve any Reactive Elements. So, H is a combination of Resistors or a Voltage Divider.

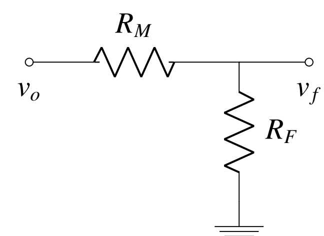

Fig. 2.1.12

$$v_f = \frac{10}{10 + 10^5} \times v_o \tag{2.1.12.1}$$
$$v_f \approx 10^{-4} v_o \tag{2.1.12.2}$$
$$\frac{v_f}{v_o} \approx 10^{-4} \tag{2.1.12.3}$$
$$H(f) = 10^{-4} \tag{2.1.12.4}$$

2.1.13. Design a Closed-Loop Transfer Function by combining both the Open-Loop and Feedback Circuits. Also draw its Equivalent Circuit
Solution:
The Closed-Loop Circuit is
The Equivalent Circuit of Closed-Loop Circuit is

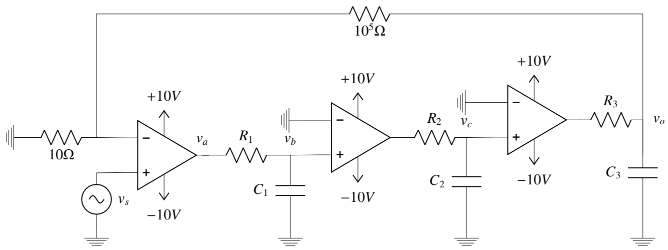

Fig. 2.1.13

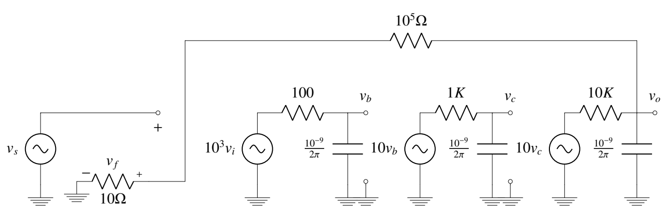

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Fig. 2.1.13

From the Equivalent Circuit Diagram,

$$G = \frac{v_o}{v_i} = \frac{10^5}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \tag{2.1.13.1}$$

$$H = \frac{v_f}{v_o} = 10^{-4} \tag{2.1.13.2}$$

The Closed-Loop Gain,

$$v_i = v_s - v_f \tag{2.1.13.3}$$

$$\frac{v_o}{G} = v_s - Hv_o \tag{2.1.13.4}$$

$$\frac{v_o}{v_s} = \frac{G}{1 + GH} \tag{2.1.13.5}$$

So, the Closed-Loop Gain,

$$T = \frac{v_o}{v_s} = \frac{10^5}{10 + \left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \tag{2.1.13.6}$$