Control Systems

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Contents

1 Circuit Design from Bode Plot

1.1. Consider the Magnitude Bode Plot and Phase Bode Plot 1.1 of Open-Loop Transfer Function of an Amplifier. Estimate the Open-Loop Transfer Function. (Assume $'A'$ as $'G'$ and $'\beta'$ as $'H'$)

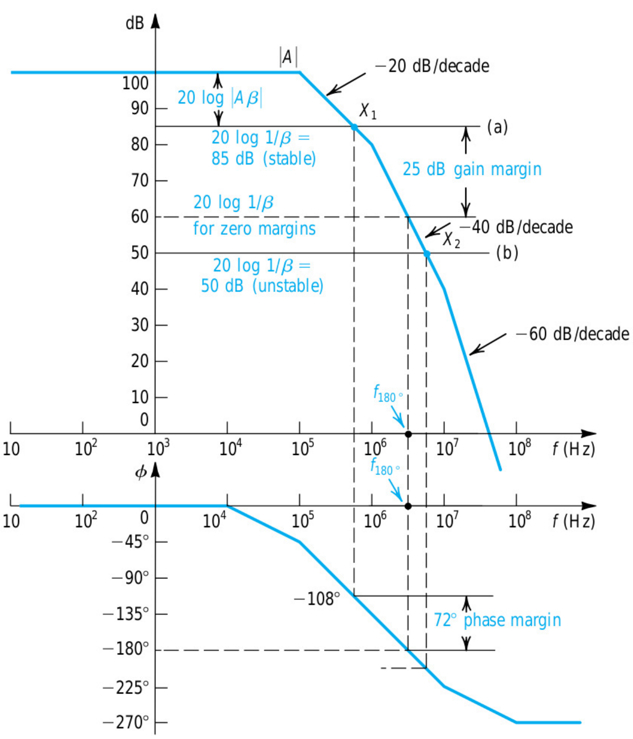

Fig. 1.1: Magnitude and Phase Bode Plot

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Solution: Let $G(f)$ be the Open-Loop Transfer Function,

$$G(f) = \begin{cases} 100 & 0 < f < 10^5 \\ 200 - 20\log(f) & 10^5 < f < 10^6 \\ 320 - 40\log(f) & 10^6 < f < 10^7 \\ 460 - 60\log(f) & 10^7 < f \end{cases} \tag{1.1.1}$$

$$\nabla G(f) = \frac{d(G(f))}{d(\log(f))} = \begin{cases} 0 & 0 < f < 10^5 \\ -20 & 10^5 < f < 10^6 \\ -40 & 10^6 < f < 10^7 \\ -60 & 10^7 < f \end{cases} \tag{1.1.2}$$

As we know that, **When a pole is encountered the slope always decreases by 20 dB/decade** and **When a zero is encountered the slope always increases by 20 dB/decade**. So, by observing Fig. 1.1 it can be concluded that we are having Poles at $f = 10^5 Hz, 10^6 Hz, 10^7 Hz$ and No Zeros.

So, the Open-Loop Transfer Function $G(f)$ is

$$G(f) = \frac{10^5}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \tag{1.1.3}$$

1.2. Calculate the Phase of Open-Loop Transfer Function.
Solution:

$$\phi(f) = -\left[\tan^{-1}\left(\frac{f}{10^5}\right) + \tan^{-1}\left(\frac{f}{10^6}\right) + \tan^{-1}\left(\frac{f}{10^7}\right)\right] \tag{1.2.1}$$

1.3. Verify (1.1.3) by plotting the Magnitude and Phase Bode Plots of $G(f)$ and comparing with (1.1.1)
Solution: See Fig. 1.3

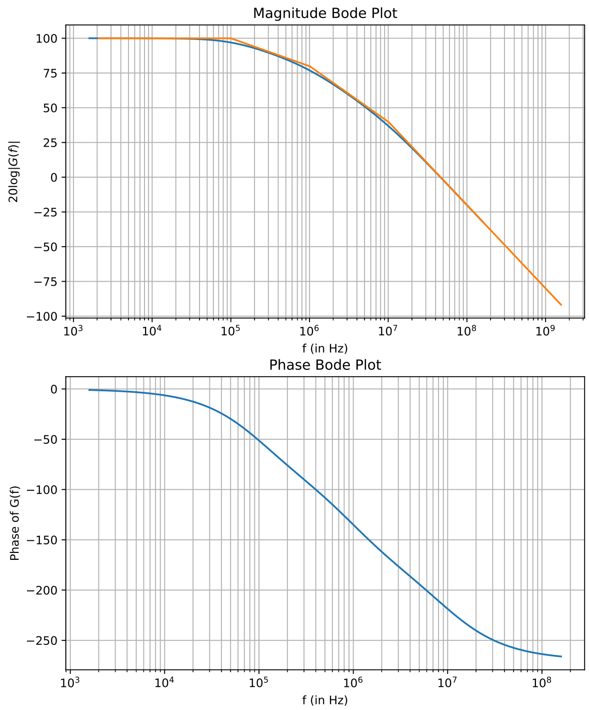

Fig. 1.3: Magnitude Bode Plot

Python Code for Bode Plot is at

```
codes/ee18btech11014/Bode_Plot.py
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1.4. Find the PM from Fig. 1.1, given that he feedback gain $H(f)$ is constant and given by

$$20\log\left(\frac{1}{H(f)}\right) = 85dB \tag{1.4.1}$$

$$\text{or, } H(f) = 5.623\times 10^{-5}. \tag{1.4.2}$$

Solution: From the figure,

$$20\log|G(f_1)| = 85dB \tag{1.4.3}$$

$$\implies 20\log|G(f_1)| = 20\log\left(\frac{1}{H(f_1)}\right) \tag{1.4.4}$$

$$\text{or, } |G(f_1)H(f_1)| = 1 \tag{1.4.5}$$

and

$$f_1 = 0.493MHz, \tag{1.4.6}$$

from (1.4.3) and (1.1.3). Also,

$$\because \angle H(f) = 0, \forall f \tag{1.4.7}$$

$$\angle G(f_1)H(f_1) = \angle G(f_1) = -108^\circ \tag{1.4.8}$$

$$\implies PM = 180^\circ - 108^\circ = 72^\circ \tag{1.4.9}$$

using (1.4.6) in (1.2.1).

1.5. Find the GM.
Solution: The crossover frequency f_π is defined as

$$\angle G(f_\pi)H(f_\pi) = 180^\circ \tag{1.5.1}$$

$$\implies \angle G(f_\pi) = 180^\circ \tag{1.5.2}$$

$$\implies f_\pi = 3.34MHz \tag{1.5.3}$$

by solving (1.2.1). From Fig. 1.1,

$$20\log|G(f_\pi)| = 60dB \tag{1.5.4}$$

$$\implies 20\log|G(f_\pi)| - 20\log\left(\frac{1}{H(f_\pi)}\right) = (60-85)\,dB \tag{1.5.5}$$

$$\implies GM = \left|20\log|G(f_\pi)H(f_\pi)|\right| = 25dB \tag{1.5.6}$$

1.6. Break the Transfer Function $G(s)$ into Simple Blocks and create a block diagram.
Solution: From (1.1.3)

$$G(s) = \frac{10^5}{\left(1+\frac{s}{2\pi 10^5}\right)\left(1+\frac{s}{2\pi 10^6}\right)\left(1+\frac{s}{2\pi 10^7}\right)} \tag{1.6.1}$$

The block diagram is available in Fig. 1.6

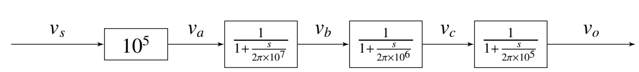

Fig. 1.6

1.7. Find the Gain of RC-Circuit in Fig. 1.7 and identify the pole location.

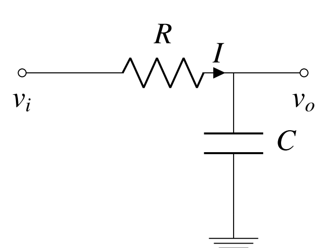

Fig. 1.7

Solution:

$$v_o = v_i \frac{\frac{1}{sc}}{R + \frac{1}{sC}} \tag{1.7.1}$$

$$\implies \frac{v_o}{v_i} = \frac{1}{1 + sCR} \tag{1.7.2}$$

Thus, there is a pole at

$$s = -\frac{1}{RC} \tag{1.7.3}$$

1.8. Design a circuit for $G(s)$.
Solution: (1.6.1) can be expressed as

$$\because G(s) = \frac{G_0}{(1 + sC_1R_1)(1 + sC_2R_2)(1 + sC_3R_3)} \tag{1.8.1}$$

where the parameters are available in Table 1.8. Choosing an OPAMP of gain G_0 and noting from (1.7.2) that each of the blocks in Fig. 1.6 can be realised through the RC circuit in Fig. 1.7 with parameters in Table 1.8, the circuit design is available in Fig. 1.8.

Circuit Element	Value
G_0	$100dB$
R_1	100Ω
R_2	$1k\Omega$
R_3	$10k\Omega$
C_1	$\frac{1}{2\pi}nF$
C_2	$\frac{1}{2\pi}nF$
C_3	$\frac{1}{2\pi}nF$

TABLE 1.8

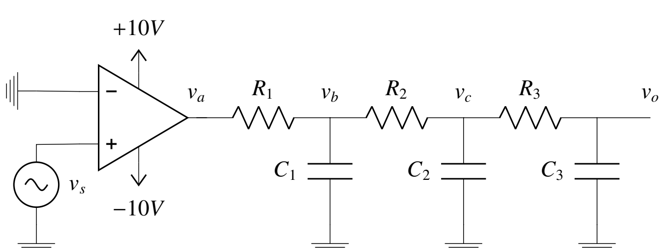

Fig. 1.8

1.9. Design a circuit for $H(s)$.
Solution: From (1.4.2), H is constant and should not involve any Reactive Elements. The simplest way to realise H is through a voltage divider as shown in Fig. 1.9. Thus,

$$H = \frac{R_F}{R_F + R_M} \tag{1.9.1}$$

with resistance values available in Table 1.9.

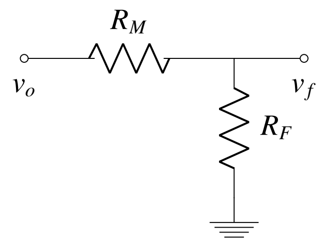

Fig. 1.9

Circuit Element	Value
R_M	$1.778 \times 10^5\Omega$
R_F	10Ω

TABLE 1.9

1.10. Find the closed loop transfer function $T(s)$ and draw the equivalent circuit.
Solution: The closed loop circuit is easily obtained from Figs. 1.8 and 1.9 as shown in Fig. 1.10

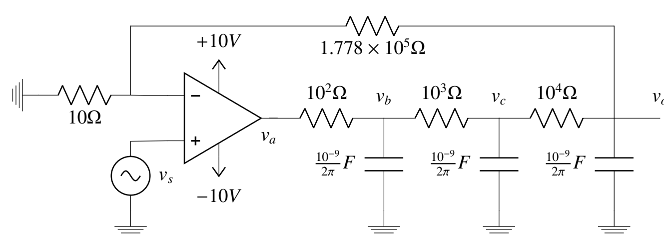

Fig. 1.10

The closed loop gain,

$$T(s) = \frac{G(s)}{1 + G(s)H} \tag{1.10.1}$$

$$= \frac{10^5}{\left(1 + \frac{s}{2\pi 10^5}\right)\left(1 + \frac{s}{2\pi 10^6}\right)\left(1 + \frac{s}{2\pi 10^7}\right) + 5.623} \tag{1.10.2}$$

2 Phase Margin

2.1. Find the frequency for which $PM = 90°$. Assume H to be constant.

Solution: $\because \angle H(f) = 1$,

$$\angle G(f_{90}) H(f_{90}) = \angle G(f_{90}) = 90^\circ - 180^\circ \tag{2.1.1}$$

$$= -90^\circ \tag{2.1.2}$$

The Bode plot in Fig. 1.1 shows that

$$|G(f)| < 1, \quad f > 10^8 \tag{2.1.3}$$

Also,

$$\tan^{-1}\left(\frac{f}{10^7}\right) \approx 0, \quad f < 10^8 \tag{2.1.4}$$

Thus, from (1.2.1) and (2.1.2),

$$\phi(f) \approx -\left[\tan^{-1}\left(\frac{f}{10^5}\right) + \tan^{-1}\left(\frac{f}{10^6}\right)\right] \tag{2.1.5}$$

$$= -90^\circ \tag{2.1.6}$$

$$\implies f_{90} = 3.162 \times 10^5 \tag{2.1.7}$$

after simplification.

2.2. Find H when the $PM = 90^\circ$.
Solution: By definition of the PM,

$$|G(f_{90}) H(f_{90})| = 1 \tag{2.2.1}$$

$$\implies |H(f_{90})| = \frac{1}{|G(f_{90})|} \tag{2.2.2}$$

From (1.1.1),

$$20\log|G(f)| = 200 - 20\log(3.162 \times 10^5) \tag{2.2.3}$$

$$= 90dB \tag{2.2.4}$$

$$\implies |G(f)| = 3.1625 \times 10^4 \tag{2.2.5}$$

$$\implies H = 3.162 \times 10^{-5} \tag{2.2.6}$$

using (2.2.2).

2.3. Design the closed loop circuit for $PM = 90^\circ$
Solution: See Fig. 2.3, where Fig. 1.9 is used for the feedback H with $R_M = 0.3162M\Omega$ and $R_F = 10\Omega$.

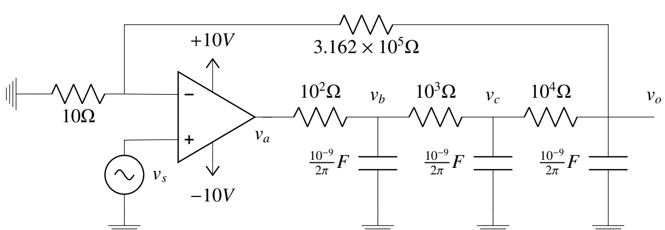

Fig. 2.3

2.4. Repeat all the above for $PM = 45^\circ$.

3 Stabilty

3.1. Discuss relation between Stability and Phase Margin.
Solution:
Assuming Loop Gain if GH. Denoting the frequency at which the magnitude of loop gain is unity by ω_1, So it can be written as

Assuming Phase-Margin = α as

$$G(j\omega_1)H = 1 \times e^{-j\theta} \tag{3.1.1}$$

$$\theta = 180^\circ - \alpha \tag{3.1.2}$$

At ω_1 the closed-loop gain is

$$T(j\omega_1) = \frac{G(j\omega_1)}{1 + G(j\omega_1)H} \tag{3.1.3}$$

$$T(j\omega_1) = \frac{(1/H)e^{-j\theta}}{1 + e^{-j\theta}} \tag{3.1.4}$$

$$|T(j\omega_1)| = \frac{1/H}{|1 + e^{-j\theta}|} \tag{3.1.5}$$

When Phase-Margin α is reduced, eventually when $\theta = 180^\circ$ system becomes Marginally Stable. And when $\theta > 180^\circ$, system becomes unstable.
So, a system is **Stable** if $\alpha > 0$, **Marginally Stable** if $\alpha = 0$ and **Unstable** if $\alpha < 0$.

3.2. If Phase of Loop Gain is ϕ at which the magnitude of loop gain is unity. Find range of ϕ for the system to be Stable, Marginally Stable and Unstable.
Solution:

$$\phi = \alpha - 180^\circ \tag{3.2.1}$$

$$\alpha = \phi + 180^\circ \tag{3.2.2}$$

For Stable,

$$\alpha > 0 \tag{3.2.3}$$

$$\phi > -180^\circ \tag{3.2.4}$$

For Marginally Stable,

$$\alpha = 0 \tag{3.2.5}$$

$$\phi = -180^\circ \tag{3.2.6}$$

For Unstable,

$$\alpha < 0 \tag{3.2.7}$$

$$\phi < -180^\circ \tag{3.2.8}$$

3.3. Find frequency at which $\phi = -180^\circ$. And Determine the region for Stability.
Solution:
H is a constant independent of Frequency. So

$$\angle G(f)H(f) = \angle G(f) \tag{3.3.1}$$

$$\angle G(f) = -\left[\tan^{-1}\left(\frac{f}{10^5}\right) + \tan^{-1}\left(\frac{f}{10^6}\right) + \tan^{-1}\left(\frac{f}{10^7}\right)\right] \tag{3.3.2}$$

As $\phi = -180^\circ$, frequency should be between $f = 10^6 Hz$ and $f = 10^7 Hz$.

$$f = 3.34 \times 10^6 Hz \tag{3.3.3}$$

So, for

- $f > 3.34 \times 10^6 Hz$, System is Unstable
- $f = 3.34 \times 10^6 Hz$, System is Marginally Stable
- $f < 3.34 \times 10^6 Hz$, System is Stable

3.4. Determine the range of H for Stability.
Solution:
At $f = 3.34 \times 10^6 Hz$,

$$G(f) = 320 - 40\log(f) \tag{3.4.1}$$

$$G = 320 - 40\log(3.34 \times 10^6) \tag{3.4.2}$$

$$G = 59dB \tag{3.4.3}$$

$$G = 896 \tag{3.4.4}$$

If Loop Gain $|G(f)H| = 1$ at $f = 3.34 \times 10^6 Hz$,

$$G = 896 \tag{3.4.5}$$

$$H = 1.11 \times 10^{-3} \tag{3.4.6}$$

If f **decreases** below $f = 3.34 \times 10^6 Hz$, G increases and the value of H at which Loop-Gain becomes unity **decreases** below $H = 1.11 \times 10^{-3}$.
So, for

- $H > 1.11 \times 10^{-3}$, System is Unstable
- $H = 1.11 \times 10^{-3}$, System is Marginally Stable
- $H < 1.11 \times 10^{-3}$, System is Stable

3.5. Verify the stablity from the value of H
Solution:
Run the following code to verify the results

```
codes/ee18btech11014/Stability.py
```