Control Systems

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Contents

***Abstract*—This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/codes
```

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1 Signal Flow Graph

2 Gain of Feedback Circuits

2.1 Estimation of Voltage Gain

2.1.1. Consider the Magnitude Bode Plot and Phase Bode Plot 2.1.1 of Open-Loop Transfer Function of an Amplifier. Estimate the Open-Loop Transfer Function. (Assume $'A'$ as $'G'$ and $'\beta'$ as $'H'$)

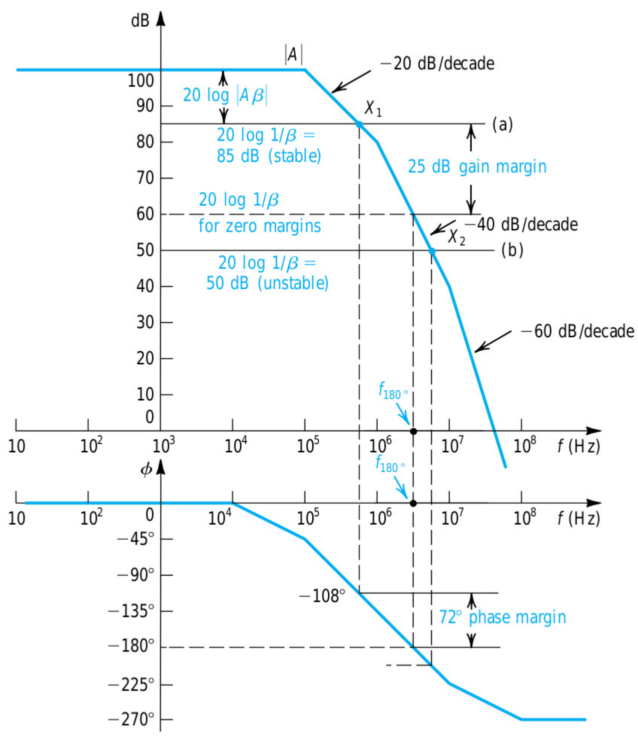

Fig. 2.1.1: Magnitude and Phase Bode Plot

Solution: Let $G(f)$ be the Open-Loop Transfer Function,

$$G(f) = \begin{cases} 100 & 0 < f < 10^5 \\ 200 - 20log(f) & 10^5 < f < 10^6 \\ 320 - 40log(f) & 10^6 < f < 10^7 \\ 460 - 60log(f) & 10^7 < f \end{cases} \tag{2.1.1.1}$$

$$\nabla G(f) = \frac{d(G(f))}{d(log(f))} = \begin{cases} 0 & 0 < f < 10^5 \\ -20 & 10^5 < f < 10^6 \\ -40 & 10^6 < f < 10^7 \\ -60 & 10^7 < f \end{cases} \tag{2.1.1.2}$$

As we know that, **When a pole is encountered the slope always decreases by 20 dB/decade** and **When a zero is encountered the slope always increases by 20 dB/decade**. So, by observing 2.1.1 it can be concluded that we are having Poles at $f = 10^5 Hz, 10^6 Hz, 10^7 Hz$ and No Zeros.

So, the Open-Loop Transfer Function $G(f)$ is

$$G(f) = \frac{10^5}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \tag{2.1.1.3}$$

2.1.2. Calculate the Phase of Open-Loop Transfer Function.
Solution:
Phase of Open-Loop Transfer Function = ϕ

$$\phi = -\left[\tan^{-1}\left(f/10^5\right) + \tan^{-1}\left(f/10^6\right) + \tan^{-1}\left(f/10^7\right)\right] \tag{2.1.2.1}$$

2.1.3. Determine the Closed-Loop Voltage Gain of the System assuming $|GH| \gg 1$ and also assuming the block diagram of Control System is 2.1.3

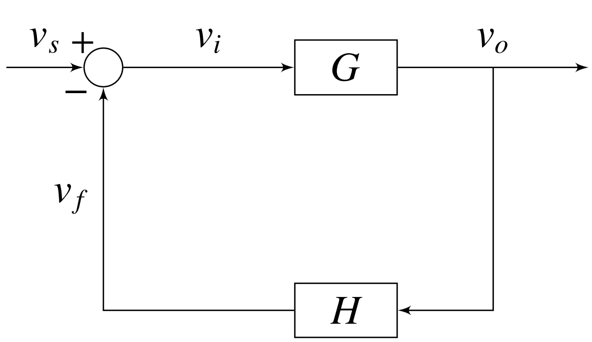

Fig. 2.1.3

Solution:
The Closed-Loop Voltage Gain of the Control System is

$$T = \frac{V_o}{V_s} = \frac{G}{1 + GH} \tag{2.1.3.1}$$

$$20\log(T) = 20\log(G) - 20\log(1 + GH) \tag{2.1.3.2}$$

Considering the assumption $|GH| \gg 1$, It can be written as

$$20\log(1 + GH) = 20\log(GH) \tag{2.1.3.3}$$

So,

$$20\log(T) = 20\log(G) - 20\log(GH) \tag{2.1.3.4}$$
$$20\log(T) = -20\log(H) \tag{2.1.3.5}$$
$$20\log(T) = 20\log(\frac{1}{H}) \tag{2.1.3.6}$$
$$T = \frac{1}{H} \tag{2.1.3.7}$$

So, The value of Closed-Loop Voltage Gain of the Control System under the assumption, $|GH| \gg 1$ is $T = \frac{1}{H}$

2.1.4. What is the value of Loop-Gain?
Solution:
The value of Loop-Gain can be calculated by the difference of 2-curves $20\log|A|$ and $20log(\frac{1}{H})$. The difference between the two curves will be

$$20\log|G| - 20\log\frac{1}{H} = 20\log|GH| \tag{2.1.4.1}$$

2.1.5. Define Phase-Margin
Solution:
Phase-Margin: The phase margin is defined as the angle in degrees by which the phase angle is smaller than -180° at the gain crossover, the gain crossover being the frequency at which the open-loop gain first reaches 1.

2.1.6. Find the frequencies for which phase margins are 90° and 45° respectively?
Solution:
Let Phase Margin be $\alpha = 90^\circ$. Then,

$$\alpha = \phi - (-180^\circ) \tag{2.1.6.1}$$
$$\phi = -180^\circ + \alpha \tag{2.1.6.2}$$
$$\phi = -90^\circ \tag{2.1.6.3}$$

So, by the definition of Phase-Margin, at $\phi = -90^\circ$, $|GH| = 1$. The value of $\phi = -90^\circ$ between poles $f = 10^5 Hz, 10^6 Hz$. Assuming the Poles are farther apart,

$$\tan^{-1}(\frac{f}{10^7}) \approx 0 \tag{2.1.6.4}$$

where $10^5 < f < 10^6$
So,

$$-\tan^{-1}\left(f/10^5\right) - \tan^{-1}\left(f/10^6\right) = -90 \tag{2.1.6.5}$$

$$\tan^{-1}\left(f/10^5\right) + \tan^{-1}\left(f/10^6\right) = 90 \tag{2.1.6.6}$$

$$\tan^{-1}\left(f/10^5\right) = 90 - \tan^{-1}\left(f/10^6\right) \tag{2.1.6.7}$$

$$\tan^{-1}\left(f/10^5\right) = \cot^{-1}\left(f/10^6\right) \tag{2.1.6.8}$$

$$\tan^{-1}\left(f/10^5\right) = \tan^{-1}\left(10^6/f\right) \tag{2.1.6.9}$$

$$f^2 = 10^{11} \tag{2.1.6.10}$$

$$f = 3.162 \times 10^5 \tag{2.1.6.11}$$

So, the approximate value of f at which Phase Margin is 90° is $f = 3.162 \times 10^5 Hz$.

Similarly let Phase Margin be $\alpha = 45^\circ$. Then,

$$\alpha = \phi - (-180^\circ) \tag{2.1.6.12}$$

$$\phi = -180^\circ + \alpha \tag{2.1.6.13}$$

$$\phi = -135^\circ \tag{2.1.6.14}$$

So, by the definition of Phase-Margin, at $\phi = -135^\circ$, $|GH| = 1$. The value of $\phi = -135^\circ$ aproximately at poles $f = 10^6 Hz$.
So, the approximate value of f at which Phase Margin is 45° is $f = 10^6$.

2.1.7. Find the minimum values of Closed-Loop Voltage Gain for which phase margins are 90° and 45° respectively
Solution:
For $\alpha = 90^\circ$,

$$f = 3.162 \times 10^5 \tag{2.1.7.1}$$

By substituting f in Open-Loop Gain $G(f)$ (assuming poles are far part),

$$G(f) = 200 - 20log(3.162 \times 10^5) \tag{2.1.7.2}$$

$$G(f) = 90dB \tag{2.1.7.3}$$

$$G = 3.1625 \times 10^4 \tag{2.1.7.4}$$

At that $f = 3.162 \times 10^5$,

$$H = \frac{1}{G} \tag{2.1.7.5}$$

$$H = 3.162 \times 10^{-5} \tag{2.1.7.6}$$

The minimum value of Closed-Loop Gain occurs at $|GH| \gg 1$ and the value of Closed-Loop Gain is $T = \frac{1}{H}$

$$T = \frac{1}{H} = 3.1625 \times 10^4 \tag{2.1.7.7}$$

So, The minimum value of Closed-Loop Gain with Phase Margin equal to $\alpha = 90^\circ$ is $T_{min} = 3.1625 \times 10^4$.

For $\alpha = 45^\circ$,

$$f = 10^6 \tag{2.1.7.8}$$

By substituting f in Open-Loop Gain $G(f)$ (assuming poles are far part),

$$G(f) = 200 - 20log(10^6) \tag{2.1.7.9}$$

$$G(f) = 80dB \tag{2.1.7.10}$$

$$G = 10^4 \tag{2.1.7.11}$$

At that $f = 10^6$,

$$H = \frac{1}{G} \tag{2.1.7.12}$$

$$H = 10^{-4} \tag{2.1.7.13}$$

The minimum value of Closed-Loop Gain occurs at $|GH| \gg 1$ and the value of Closed-Loop Gain is $T = \frac{1}{H}$

$$T = \frac{1}{H} = 10^4 \tag{2.1.7.14}$$

So, The minimum value of Closed-Loop Gain with Phase Margin equal to $\alpha = 45^\circ$ is $T_{min} = 10^4$.

2.1.8. Break the Transfer Function $G(f)$ into Simple Blocks and Create a Block Diagram for $G(f)$.
Solution:

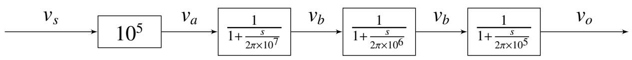

Fig. 2.1.8

2.1.9. Find the Gain of RC-Circuit shown below 2.1.9 and also identify the location of Poles.

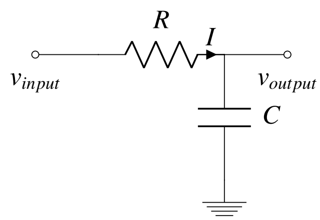

Fig. 2.1.9

Solution:

$$I = \frac{v_{input}}{R + \frac{1}{Cs}} \tag{2.1.9.1}$$

$$v_{output} = I \times \frac{1}{Cs} \tag{2.1.9.2}$$

$$v_{output} = \frac{v_{input} \times \frac{1}{Cs}}{R + \frac{1}{Cs}} \tag{2.1.9.3}$$

$$\frac{v_{output}}{v_{input}} = \frac{1}{RCs + 1} \tag{2.1.9.4}$$

$$s = j2\pi f \tag{2.1.9.5}$$

$$Gain = \frac{v_{output}}{v_{input}} = \frac{1}{j2\pi RCf + 1} \tag{2.1.9.6}$$

So, there is a Pole at frequency $f = \frac{1}{2\pi RC}$ for the Transfer Function of Gain.

2.1.10. Find the Gain of Operational Amplifier. The circuit diagram of Equivalent Circuit is 2.1.10.

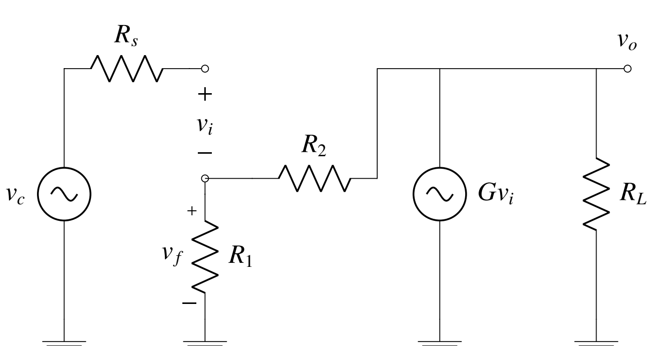

Fig. 2.1.10

Solution:
Applying KVL and KCL,

$$v_o = Gv_i \tag{2.1.10.1}$$

As no current flows through R_s,

$$v_i = v_c - v_f \tag{2.1.10.2}$$

$$v_f = \frac{R_1}{R_1 + R_2} v_o \tag{2.1.10.3}$$

$$v_i = \frac{v_o}{G} \tag{2.1.10.4}$$

$$\frac{v_o}{G} = v_c - \frac{R_1}{R_1 + R_2} v_o \tag{2.1.10.5}$$

$$\frac{v_o}{v_c} = \frac{G}{1 + G\frac{R_1}{R_1+R_2}} \tag{2.1.10.6}$$

So, Gain of the Circuit is $\frac{G}{1+G\frac{R_1}{R_1+R_2}}$

2.1.11. Design a Circuit Model that follows the Transfer Function $G(f)$
Solution:
Our Design for Modelling the Transfer Function is based on Poles of RC-Circuit and Gain of Operational Amplifier.

So, the Circuit Diagram is,

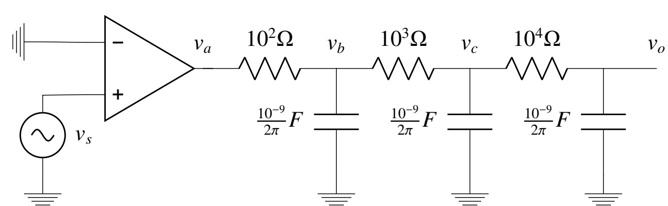

Fig. 2.1.11

Assuming, Open-Loop Gain of Operational Amplifier is 10^5 and also assuming Operational Amplifier doesnt have any Poles.
Equivalent Circuit of the circuit is

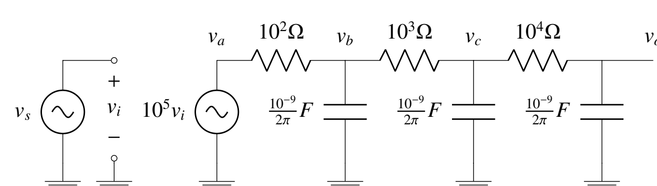

Fig. 2.1.11

The cascade of RC Circuits are used to introduce poles in the circuit and Op-Amp are used to achieve the Gain required.

At the Operational Amplifier,

$$v_i = v_s \tag{2.1.11.1}$$
$$v_a = 10^5 v_i \tag{2.1.11.2}$$
$$v_a = 10^5 v_s \tag{2.1.11.3}$$

At the first RC-Circuit,

$$2\pi RC = 10^{-7} \tag{2.1.11.4}$$
$$v_b = \frac{v_a}{1 + j\frac{f}{10^7}} \tag{2.1.11.5}$$
$$v_b = \frac{10^5 v_i}{1 + j\frac{f}{10^7}} \tag{2.1.11.6}$$

At the second RC-Circuit,

$$2\pi RC = 10^{-6} \tag{2.1.11.7}$$
$$v_c = \frac{v_b}{1 + j\frac{f}{10^6}} \tag{2.1.11.8}$$
$$v_c = \frac{10^5 v_i}{(1 + j\frac{f}{10^6})(1 + j\frac{f}{10^7})} \tag{2.1.11.9}$$

At the third RC-Circuit,

$$2\pi RC = 10^{-5} \tag{2.1.11.10}$$
$$v_o = \frac{v_c}{1 + j\frac{f}{10^5}} \tag{2.1.11.11}$$

$$v_o = \frac{10^5 v_i}{(1 + j\frac{f}{10^5})(1 + j\frac{f}{10^6})(1 + j\frac{f}{10^7})} \tag{2.1.11.12}$$

The RC Circuits introduces poles at $f = 10^7 Hz, 10^6 Hz, 10^5 Hz$ respectively from left to right and Op-Amp introduced a Gain = 10^5. So, the value of v_o is

$$v_o = \frac{10^5 v_i}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \tag{2.1.11.13}$$

So, Open-Loop Gain is

$$G = \frac{10^5}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \tag{2.1.11.14}$$

2.1.12. Design a Circuit Model that follows the Feedback Transfer Function $H(f)$
Solution:
On Bode Plot is H is independent of frequency. So, H should not involve any Reactive Elements. So, H is a combination of Resistors or a Voltage Divider.

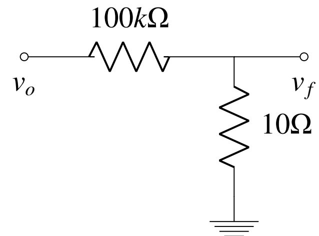

Fig. 2.1.12

$$v_f = \frac{10}{10 + 10^5} \times v_o \tag{2.1.12.1}$$
$$v_f \approx 10^{-4} v_o \tag{2.1.12.2}$$
$$\frac{v_f}{v_o} \approx 10^{-4} \tag{2.1.12.3}$$
$$H(f) = 10^{-4} \tag{2.1.12.4}$$

2.1.13. Design a Closed-Loop Transfer Function by combining both the Open-Loop and Feedback Circuits. Also draw its Equivalent Circuit
Solution:
The Closed-Loop Circuit is

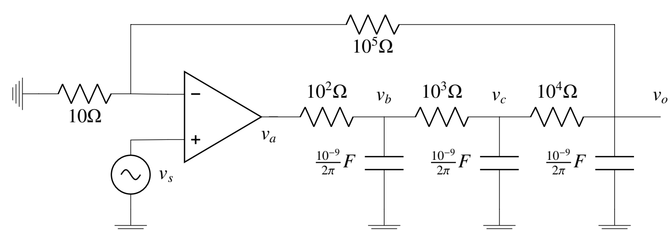

Fig. 2.1.13

The Equivalent Circuit of Closed-Loop Circuit is

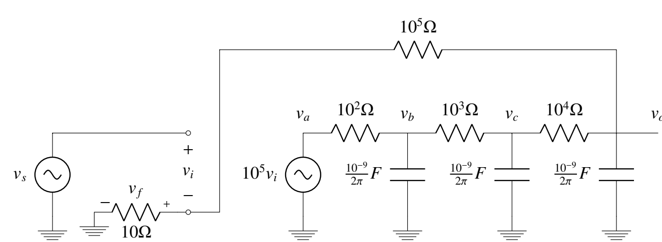

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Fig. 2.1.13

From the Equivalent Circuit Diagram,

$$G = \frac{v_o}{v_i} = \frac{10^5}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \tag{2.1.13.1}$$

$$H = \frac{v_f}{v_o} = 10^{-4} \tag{2.1.13.2}$$

The Closed-Loop Gain,

$$v_i = v_s - v_f \tag{2.1.13.3}$$

$$\frac{v_o}{G} = v_s - Hv_o \tag{2.1.13.4}$$

$$\frac{v_o}{v_s} = \frac{G}{1 + GH} \tag{2.1.13.5}$$

So, the Closed-Loop Gain,

$$T = \frac{v_o}{v_s} = \frac{10^5}{10 + \left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \tag{2.1.13.6}$$

3 Bode Plot

4 Second order System

5 Routh Hurwitz Criterion

6 State-Space Model

7 Nyquist Plot

8 Compensators

9 Gain Margin

10 Phase Margin

11 Oscillator

12 Root Locus

13 Polar Plot

14 PID Controller