Control Systems

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Contents

***Abstract*—This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
  control/codes
```

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1 Signal Flow Graph

2 Gain of Feedback Circuits

2.1 Estimation of Voltage Gain

2.1.1. Consider the Magnitude Bode Plot and Phase Bode Plot 2.1.1 of Open-Loop Transfer Function of an Amplifier. Estimate the Open-Loop Transfer Function. (Assume $'A'$ as $'G'$ and $'\beta'$ as $'H'$)

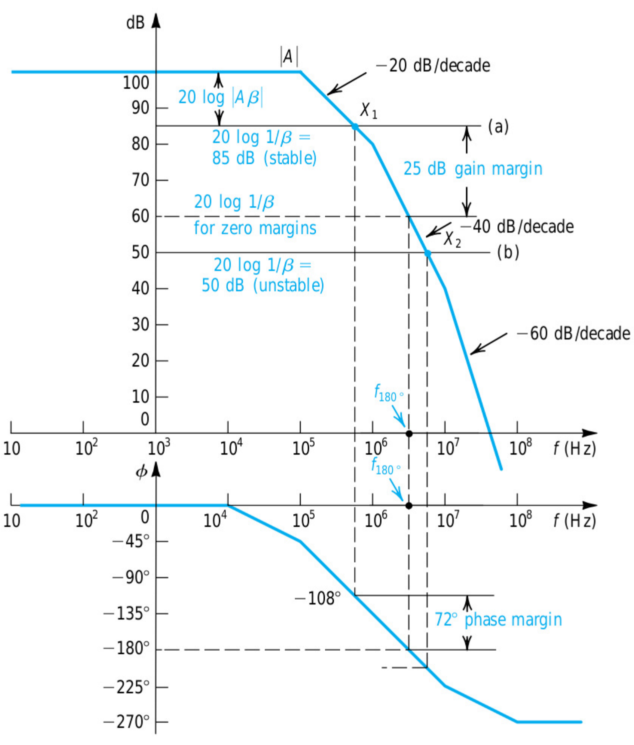

Fig. 2.1.1: Magnitude and Phase Bode Plot

Solution: Let $G(f)$ be the Open-Loop Transfer Function,

$$G(f) = \begin{cases} 100 & 0 < f < 10^5 \\ 200 - 20log(f) & 10^5 < f < 10^6 \\ 320 - 40log(f) & 10^6 < f < 10^7 \\ 460 - 60log(f) & 10^7 < f \end{cases} \tag{2.1.1.1}$$

$$\nabla G(f) = \frac{d(G(f))}{d(log(f))} = \begin{cases} 0 & 0 < f < 10^5 \\ -20 & 10^5 < f < 10^6 \\ -40 & 10^6 < f < 10^7 \\ -60 & 10^7 < f \end{cases} \quad (2.1.1.2)$$

As we know that, **When a pole is encountered the slope always decreases by 20 dB/decade** and **When a zero is encountered the slope always increases by 20 dB/decade**. So, by observing 2.1.1 it can be concluded that we are having Poles at $f = 10^5 Hz, 10^6 Hz, 10^7 Hz$ and No Zeros.

So, the Open-Loop Transfer Function $G(f)$ is

$$G(f) = \frac{10^5}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \quad (2.1.1.3)$$

2.1.2. Calculate the Phase of Open-Loop Transfer Function.
Solution:
Phase of Open-Loop Transfer Function = ϕ

$$\phi = -\left[\tan^{-1}\left(f/10^5\right) + \tan^{-1}\left(f/10^6\right) + \tan^{-1}\left(f/10^7\right)\right] \quad (2.1.2.1)$$

2.1.3. Determine the Closed-Loop Voltage Gain of the System assuming $|GH| \gg 1$ and also assuming the block diagram of Control System is 2.1.3

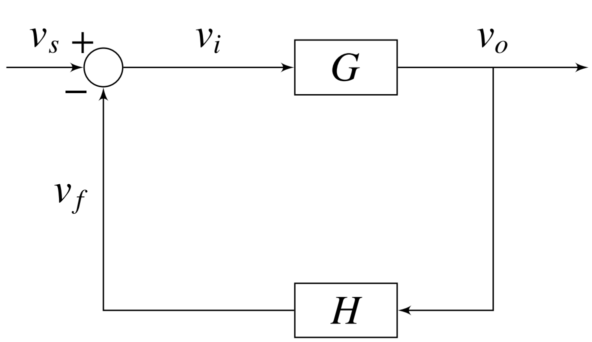

Fig. 2.1.3

Solution:
The Closed-Loop Voltage Gain of the Control System is

$$T = \frac{V_o}{V_s} = \frac{G}{1+GH} \quad (2.1.3.1)$$

$$20\log(T) = 20\log(G) - 20\log(1+GH) \quad (2.1.3.2)$$

Considering the assumption $|GH| \gg 1$, It can be written as

$$20\log(1+GH) = 20\log(GH) \quad (2.1.3.3)$$

So,

$$20\log(T) = 20\log(G) - 20\log(GH) \quad (2.1.3.4)$$

$$20\log(T) = -20\log(H) \quad (2.1.3.5)$$

$$20\log(T) = 20\log(\frac{1}{H}) \quad (2.1.3.6)$$

$$T = \frac{1}{H} \quad (2.1.3.7)$$

So, The value of Closed-Loop Voltage Gain of the Control System under the assumption, $|GH| \gg 1$ is $T = \frac{1}{H}$

2.1.4. What is the value of Loop-Gain?
Solution:
The value of Loop-Gain can be calculated by the difference of 2-curves $20\log|A|$ and $20log(\frac{1}{H})$. The difference between the two curves will be

$$20\log|G| - 20\log\frac{1}{H} = 20\log|GH| \quad (2.1.4.1)$$

2.1.5. Define Phase-Margin
Solution:
Phase-Margin: The phase margin is defined as the angle in degrees by which the phase angle is smaller than $-180°$ at the gain crossover, the gain crossover being the frequency at which the open-loop gain first reaches 1.

2.1.6. Find the frequencies for which phase margins are 90° and 45° respectively?
Solution:
Let Phase Margin be $\alpha = 90°$. Then,

$$\alpha = \phi - (-180°) \quad (2.1.6.1)$$

$$\phi = -180° + \alpha \quad (2.1.6.2)$$

$$\phi = -90° \quad (2.1.6.3)$$

So, by the definition of Phase-Margin, at $\phi = -90^\circ$, $|GH| = 1$. The value of $\phi = -90^\circ$ between poles $f = 10^5 Hz, 10^6 Hz$. Assuming the Poles are farther apart,

$$\tan^{-1}(\frac{f}{10^7}) \approx 0 \quad (2.1.6.4)$$

where $10^5 < f < 10^6$
So,

$$-\tan^{-1}\left(f/10^5\right) - \tan^{-1}\left(f/10^6\right) = -90 \quad (2.1.6.5)$$

$$\tan^{-1}\left(f/10^5\right) + \tan^{-1}\left(f/10^6\right) = 90 \quad (2.1.6.6)$$

$$\tan^{-1}\left(f/10^5\right) = 90 - \tan^{-1}\left(f/10^6\right) \quad (2.1.6.7)$$

$$\tan^{-1}\left(f/10^5\right) = \cot^{-1}\left(f/10^6\right) \quad (2.1.6.8)$$

$$\tan^{-1}\left(f/10^5\right) = \tan^{-1}\left(10^6/f\right) \quad (2.1.6.9)$$

$$f^2 = 10^{11} \quad (2.1.6.10)$$

$$f = 3.162 \times 10^5 \quad (2.1.6.11)$$

So, the approximate value of f at which Phase Margin is 90° is $f = 3.162 \times 10^5 Hz$.

Similarly let Phase Margin be $\alpha = 45^\circ$. Then,

$$\alpha = \phi - (-180^\circ) \quad (2.1.6.12)$$

$$\phi = -180^\circ + \alpha \quad (2.1.6.13)$$

$$\phi = -135^\circ \quad (2.1.6.14)$$

So, by the definition of Phase-Margin, at $\phi = -135^\circ$, $|GH| = 1$. The value of $\phi = -135^\circ$ aproximately at poles $f = 10^6 Hz$.
So, the approximate value of f at which Phase Margin is 45° is $f = 10^6$.

2.1.7. Find the minimum values of Closed-Loop Voltage Gain for which phase margins are 90° and 45° respectively
Solution:
For $\alpha = 90^\circ$,

$$f = 3.162 \times 10^5 \quad (2.1.7.1)$$

By substituting f in Open-Loop Gain $G(f)$ (assuming poles are far part),

$$G(f) = 200 - 20log(3.162 \times 10^5) \quad (2.1.7.2)$$

$$G(f) = 90dB \quad (2.1.7.3)$$

$$G = 3.1625 \times 10^4 \quad (2.1.7.4)$$

At that $f = 3.162 \times 10^5$,

$$H = \frac{1}{G} \quad (2.1.7.5)$$

$$H = 3.162 \times 10^{-5} \quad (2.1.7.6)$$

The minimum value of Closed-Loop Gain occurs at $|GH| \gg 1$ and the value of Closed-Loop Gain is $T = \frac{1}{H}$

$$T = \frac{1}{H} = 3.1625 \times 10^4 \quad (2.1.7.7)$$

So, The minimum value of Closed-Loop Gain with Phase Margin equal to $\alpha = 90^\circ$ is $T_{min} = 3.1625 \times 10^4$.

For $\alpha = 45^\circ$,

$$f = 10^6 \quad (2.1.7.8)$$

By substituting f in Open-Loop Gain $G(f)$ (assuming poles are far part),

$$G(f) = 200 - 20log(10^6) \quad (2.1.7.9)$$

$$G(f) = 80dB \quad (2.1.7.10)$$

$$G = 10^4 \quad (2.1.7.11)$$

At that $f = 10^6$,

$$H = \frac{1}{G} \quad (2.1.7.12)$$

$$H = 10^{-4} \quad (2.1.7.13)$$

The minimum value of Closed-Loop Gain occurs at $|GH| \gg 1$ and the value of Closed-Loop Gain is $T = \frac{1}{H}$

$$T = \frac{1}{H} = 10^4 \quad (2.1.7.14)$$

So, The minimum value of Closed-Loop Gain with Phase Margin equal to $\alpha = 45^\circ$ is $T_{min} = 10^4$.

2.1.8. Find the Gain of RC-Circuit shown below 2.1.8 and also identify the location of Poles.

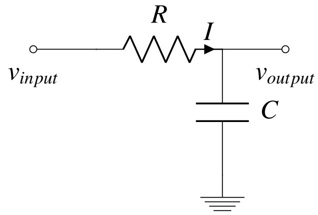

Fig. 2.1.8

Solution:

$$I = \frac{v_{input}}{R + \frac{1}{Cs}} \tag{2.1.8.1}$$

$$v_{output} = I \times \frac{1}{Cs} \tag{2.1.8.2}$$

$$v_{output} = \frac{v_{input} \times \frac{1}{Cs}}{R + \frac{1}{Cs}} \tag{2.1.8.3}$$

$$\frac{v_{output}}{v_{input}} = \frac{1}{RCs + 1} \tag{2.1.8.4}$$

$$s = j2\pi f \tag{2.1.8.5}$$

$$Gain = \frac{v_{output}}{v_{input}} = \frac{1}{j2\pi RCf + 1} \tag{2.1.8.6}$$

So, there is a Pole at frequency $f = \frac{1}{2\pi RC}$ for the Transfer Function of Gain.

2.1.9. Find the Gain of Small-Signal of N-MOSFET. The circuit diagram of Small-Signal Model is 2.1.9.

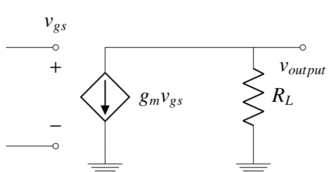

Fig. 2.1.9

Solution:

$$v_{output} = -g_m v_{gs} R_L \tag{2.1.9.1}$$

So, Gain of the above circuit is $-g_m v_{gs} R_L$.

2.1.10. Design a Circuit Model that follows the Transfer Function $G(f)$

Solution:

Our Design for Modelling the Transfer Function is based on Poles of RC-Circuit and Gain of N-MOSFET. The cascade of RC Circuits are used to introduce poles in the circuit and N-MOSFETs are used to achieve the Gain required.

So, the Circuit Diagram is,

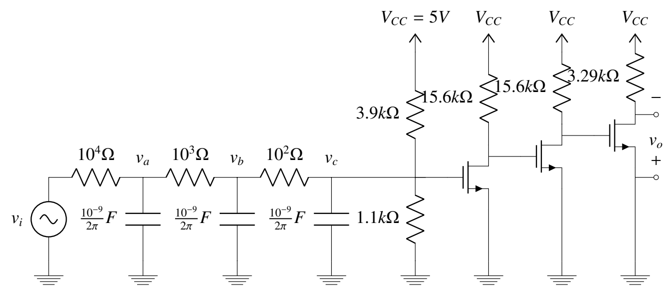

Fig. 2.1.10

Assuming, N-MOSFETs are identical and properties of N-MOSFET are,

$$\mu_n C_{ox} = 100\mu A/V^2 \tag{2.1.10.1}$$

$$\frac{W}{L} = 500 \tag{2.1.10.2}$$

$$V_T = 1V \tag{2.1.10.3}$$

V_{CC} is divided across $4.9\times10^3\Omega$ and $1.1\times10^3\Omega$

$$V_{GS} = 1.1V \tag{2.1.10.4}$$

Small Signal Parameter g_m,

$$g_m = \mu_n C_{ox} \frac{W}{L}(V_{GS} - V_{TH}) \tag{2.1.10.5}$$

$$g_m = 5 \times 10^{-3}\Omega^{-1} \tag{2.1.10.6}$$

$$I_D = \frac{g_m}{2}(V_{GS} - V_{TH}) \tag{2.1.10.7}$$

$$I_D = 2.5 \times 10^{-4} A \tag{2.1.10.8}$$

Small Signal Model of the circuit is

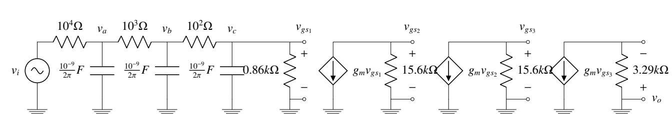

Fig. 2.1.10

At the first RC-Circuit,

$$2\pi RC = 10^{-5} \tag{2.1.10.9}$$

$$v_a = \frac{v_i}{1 + j\frac{f}{10^5}} \tag{2.1.10.10}$$

At the second RC-Circuit,

$$2\pi RC = 10^{-6} \quad (2.1.10.11)$$

$$v_b = \frac{v_a}{1 + j\frac{f}{10^6}} \quad (2.1.10.12)$$

$$v_b = \frac{v_i}{(1 + j\frac{f}{10^5})(1 + j\frac{f}{10^6})} \quad (2.1.10.13)$$

At the third RC-Circuit,

$$2\pi RC = 10^{-7} \quad (2.1.10.14)$$

$$v_c = \frac{v_b}{1 + j\frac{f}{10^7}} \quad (2.1.10.15)$$

$$v_{gs_1} = v_c \quad (2.1.10.16)$$

$$v_{gs_1} = \frac{v_i}{(1 + j\frac{f}{10^5})(1 + j\frac{f}{10^6})(1 + j\frac{f}{10^7})} \quad (2.1.10.17)$$

The RC Circuits introduces poles at $f = 10^5 Hz, 10^6 Hz, 10^7 Hz$ respectively from left to right. So, the value of v_{gs_1} is

$$v_{gs_1} = \frac{v_i}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \quad (2.1.10.18)$$

The MOSFETs used in the circuit are Self-Biased with $V_{GS} = 1.1V$ and $I_D = 2.5 \times 10^{-4} A$ for all 3 N-MOSFETs. 3 N-MOSFETs are used to achieve a large gain of 10^5. At 3rd N-MOSFET, Terminals of output are swaped, inorder to compensate for the negetive sign of gain.

At the first N-MOSFET,

$$v_{gs_2} = -g_m v_{gs_1} \times 15.6 \times 10^3 \quad (2.1.10.19)$$

$$v_{gs_2} = -78 \times v_{gs_1} \quad (2.1.10.20)$$

$$v_{gs_2} = \frac{-78 \times v_i}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \quad (2.1.10.21)$$

Similarly, at the second N-MOSFET,

$$v_{gs_3} = -g_m v_{gs_2} \times 15.6 \times 10^3 \quad (2.1.10.22)$$

$$v_{gs_3} = -78 \times v_{gs_2} \quad (2.1.10.23)$$

$$v_{gs_3} = \frac{6084 \times v_i}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \quad (2.1.10.24)$$

Similarly, at the third N-MOSFET,

$$-v_o = -g_m v_{gs_3} \times 3.29 \times 10^3 \quad (2.1.10.25)$$

$$v_o = 16.45 \times v_{gs_3} \quad (2.1.10.26)$$

$$v_o = \frac{10^5 \times v_i}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \quad (2.1.10.27)$$

$$\frac{v_o}{v_i} = \frac{10^5}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \quad (2.1.10.28)$$

$$G(f) = \frac{10^5}{\left(1 + j\frac{f}{10^5}\right)\left(1 + j\frac{f}{10^6}\right)\left(1 + j\frac{f}{10^7}\right)} \quad (2.1.10.29)$$

2.1.11. Design a Circuit Model that follows the Transfer Function $H(f)$

Solution:

On Bode Plot is H is independent of frequency. So, H should not involve any Reactive Elements. So, H is a combination of Resistors or a Voltage Divider.

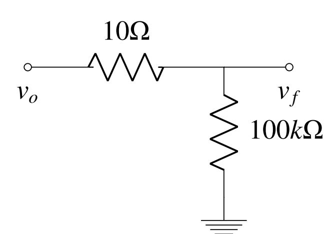

Fig. 2.1.11

$$v_f = \frac{10}{10 + 10^5} \times v_o \quad (2.1.11.1)$$

$$v_f \approx 10^{-4} v_o \quad (2.1.11.2)$$

$$\frac{v_f}{v_o} \approx 10^{-4} \quad (2.1.11.3)$$

$$H(f) = 10^{-4} \quad (2.1.11.4)$$

3 Bode Plot

4 Second order System

5 Routh Hurwitz Criterion

6 State-Space Model

7 Nyquist Plot

8 Compensators

9 Gain Margin

10 Phase Margin

11 Oscillator

12 Root Locus

13 Polar Plot

14 PID Controller