Let an initial value problem be specified as follows.

$$y' = f(t, y), \quad y(t_0) = y_0$$

In words, what this means is that the rate at which y changes is a function of y and of t (time). At the start, time is t_0 and y is y_0.

The simplest RungeKutta method is the (forward) Euler method, given by the formula

$$y_{n+1} = y_n + hf(t_n, y_n).$$

An example of a second-order method with two stages is provided by the midpoint method

$$y_{n+1} = y_n + hf\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(t_n, y_n)\right).$$

One member of the family of RungeKutta methods is so commonly used that it is often referred to as "RK4", "classical Runge-Kutta method" or simply as "the RungeKutta method".

The RK4 method for this problem is given by the following equations:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{1}$$

$$t_{n+1} = t_n + h \tag{2}$$

$$\tag{3}$$

where y_{n+1} is the RK4 approximation of $y(t_{n+1})$, and

$$k_1 = hf(t_n, y_n) \tag{4}$$

$$k_2 = hf\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right) \tag{5}$$

$$k_3 = hf\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right) \tag{6}$$

$$k_4 = hf(t_n + h, y_n + k_3) \tag{7}$$

Solve the following equations.

$$H = \frac{1}{2}(P_x^2 + P_y^2) + v(x, y)$$

$$\frac{dx}{dt} = \frac{\partial H}{\partial P_x} = p_x, \qquad \frac{dy}{dt} = \frac{\partial H}{\partial P_y} = p_y$$

$$\frac{dP_x}{dt} = -\frac{\partial H}{\partial x} = -\frac{\partial v}{\partial x}, \qquad \frac{dP_y}{dt} = \frac{\partial H}{\partial y} = -\frac{\partial v}{\partial y}$$

where 1. $v(x, y) = 0.5(x^2 + y^2)$ 2. $v(x, y) = \sqrt{(x^2 + y^2)}$
3. $v(x, y) = 0.5(x^2 + y^2) + x^2y - \frac{1}{3}y^3$