Examples and Pathologies

This will be our last lecture on the Hilbert Scheme. Here we will state the existence of several variations of $\mathrm{Hilb}_r^{P(t)}$ (whose construction is "similar"). We then turn to the concrete example of the Hilbert scheme of twisted cubics, and a brief tour of Mumford's pathological example.

Extended Example: Hilbert Scheme of Twisted Cubics:

We first review some classical geometry. A twisted cubic is, by definition, a rational projectively normal curve in $\mathbb{P}^3$. Alternatively (via Riemann-Roch) it is the only class of smooth cubic in $\mathbb{P}^3$, contained in no hyperplanes. Since the image of the 3-Uple embedding $\nu_3 : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$ is one, we see that any twisted cubic differs from $\nu_3(\mathbb{P}^1)$ by an automorphism of $\mathbb{P}^3$.

We first show that twisted cubics in $\mathbb{P}^3$ are parametrized by a 12-dimensional noncompact variety. Let us first assume that such a space H_0 exists. Fix some distinguished twisted cubic $C \in H_0$. Define a map $PGL(4, \mathbb{C}) \longrightarrow H_0$ by $g \mapsto g(C)$, and this is clearly surjective. There is a dense open set where this map is flat, and so on this set, the dimension is $\dim(U) = \dim(PGL(4, \mathbb{C})) - \dim(\mathrm{Stab}(C))$. So we need to identify $\mathrm{Stab}(C)$. This is all projective automorphisms fixing C, also known as the projective motions of C. Classically this is known to be $PGL(2, \mathbb{C})$, hence $\dim(U) = 12$. Since this is a dense open set, $\dim(H_0) = 12$ (note: H_0 is the surjective image