RESEARCH STATEMENT

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Contents

1. Introduction

My primary area of interest is in algebraic geometry. Roughly speaking, algebraic geometry is the study of solution sets to multiple polynomial equations in several variables (called algebraic varieties). Since polynomial rings are so well understood, it is natural to bring to bear many algebraic techniques to tackle geometric questions about varieties, and vice versa.

More generally, the detailed study of rings naturally involves the study of modules over them and their corresponding homological properties. In algebraic geometry, the correct notion of module is that of a (quasi-)coherent sheaf. The homological properties of coherent sheaves in many situations is effectively captured in a specific category $\mathrm{D}^b(X)$, which we call the bounded derived category of X. To construct this category, one first considers the (abelian) category of coherent sheaves $\operatorname{coh} X$[1] and then pass to the category of bounded cochain complexes $\mathrm{Ch}^b(\operatorname{coh} X)$. Roughly speaking, one then forms $\mathrm{D}^b(X)$ by formally "inverting" quasi-isomorphisms (morphisms which induce an isomorphism on cohomology). The result is no longer an abelian category, but has a different structure known as a triangulation.

The derived category was originally discovered by Grothendieck and his student, Verdier, as a natural setting for Grothendieck's general duality theory. However, since its discovery, the derived category has become an interesting object to study on its own. One of the most famous examples of this is Kontsevich's homological mirror symmetry [Kon95], but besides this the study of $\mathrm{D}^b(X)$ has resulted in a very large number of (conjectural) connections to the intrinsic geometry of the variety X, as well as to other fields in mathematics [Kuz16]. My interests in algebraic geometry lie in the study of this category, as well as its connections to birational geometry, enumerative geometry, geometric representation theory, homological mirror symmetry, and moduli theory.

[1]If the variety is affine, i.e. sits as a subset of $\mathbb{C}^n$, then $\operatorname{coh} X$ is the category of finitely generated modules over the ring of functions.

Specifically, an aspect of the theory which has, up until somewhat recently, been mostly ignored is the case when X is singular.[2] However many of the basic tools are now in place for its detailed study. This is more then a question of internal interest, as singular varieties arise naturally, see for example, the discussion in [Kaw17]. Hence it is important to understand the derived categories of singular varieties.

The major area of my work so-far concentrates on understanding $\mathrm{D}^b(X)$ when X has only Cohen-Macaulay singularities. My first project is to understand to what extent $\mathrm{D}^b(X)$ determines X. In other words, if we know $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ as triangulated categories, is it true that $X \cong Y$? When working with non-singular projective varieties, the answer is tied to the intrinsic geometry of X. Namely if ω_X[3] is ample or anti-ample, then the famous "reconstruction theorem" of Bondal and Orlov in [BO01] states that the derived category is a complete invariant. In [Spe20b] I showed that this result is still true for curves even when the curve has arbitrary singularities.

My second project involves the notion of a semiorthogonal decomposition (see 3.1 for the definition). This, written as $\mathrm{D}^b(X) = \langle \mathcal{A}_1, ..., \mathcal{A}_n \rangle$, decomposes the derived category into smaller, more manageable categories. It has been observed through many examples that the existence of certain semiorthogonal decompositions has enormous consequences for the variety in question [Kuz16]. A part of understanding these decompositions is to know when derived categories of varieties are indecomposable, that is, they admit no nontrivial semiorthogonal decompositions. Results of Kawatani and Okawa in [KO15] provide the best criteria to date, namely if the canonical bundle ω_X has "enough" global sections, then the derived category is indecomposable. My second project is an effort to understand semiorthogonal decompositions in the singular case.

2. Reconstruction of singular curves

Before detailing the first result, we need some context. My result shows that any equivalence between derived categories of curves induces an isomorphism of the two curves. The strategy is simple; from the functor $F : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$, produce an as-explicit-as-possible isomorphism $f : X \to Y$. This is far from possible in general, but there are a class of functors, called integral functors, where this is sometimes possible.

Given a functor $F : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$, it is a desirable property to be represented by an object on the product. Consider the diagram:

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y, \end{tikzcd}$$

with π_X and π_Y the projections. Given some object $\mathcal{K}$ in $\mathrm{D}^b(X \times Y)$, called the kernel, we ask if F is isomorphic to the triangulated functor defined by the pull-multiply-push formula:

$$\Phi_{\mathcal{K}}^{X \to Y}(-) = \mathbf{R}\pi_{Y*}(\pi_X^*(-) \overset{\mathbf{L}}{\otimes} \mathcal{K}).$$

If so, we say that F is an integral functor with kernel $\mathcal{K}$. If it is also an equivalence, we say that it is a Fourier-Mukai functor.

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[2] X is said to be singular when the Jacobian matrix of its defining polynomials has less then its maximum possible rank.

[3] ω_X is the so-called "canonical bundle". It is defined as the top exterior power of the cotangent bundle. In the singular setting, the replacement is the dualizing sheaf, which is typically denoted by the same symbol.

For algebraic geometers, integral functors have taken center stage as the functors of primary interest, due both to utility and necessity, as seminal results of Lunts and Orlov show that any fully faithful functor $\mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ between the bounded derived categories of projective varieties with a a right adjoint is an integral functor [LO10]. In particular, any equivalence is of this form.

Returning to the question of reconstruction; in the singular setting, the reconstruction theorem has been generalized to Gorenstein varieties with ample or anti-ample dualizing bundle [Bal11, SdSSdS12]. The next most general type of singularity is that of Cohen-Macaulay type, and in fact in dimension one, singularities can be at worst Cohen-Macaulay. My contribution in [Spe20b] completes the reconstruction problem in dimension one by proving a reconstruction theorem for Cohen-Macaulay curves. More explicitly, a simple corollary to my work is the following.

Theorem 2.1. *Let X and Y be integral complex projective varieties such that $\dim X = 1$. Then $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ if and only if $X \cong Y$.*

Note that we make no assumptions on the possible singularities.

While this is interesting in its own right, the method of proof also yielded a few useful technical lemmas which are more widely applicable. For example, an extension/restriction type theorem for Fourier-Mukai functors:

Lemma 2.1. *Let X and Y be two projective varieties over the field k. Then given $\mathcal{K} \in \mathrm{D}(\mathrm{Qcoh}(X \times Y))$, the following are equivalent:*

(1) $\Phi_{\mathcal{K}}^{X\to Y} : \mathrm{D}(\mathrm{Qcoh}\, X) \xrightarrow{\sim} \mathrm{D}(\mathrm{Qcoh}\, Y)$,
(2) $\Phi_{\mathcal{K}}^{X\to Y} : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$,
(3) $\Phi_{\mathcal{K}}^{X\to Y} : \mathrm{Perf}\, X \xrightarrow{\sim} \mathrm{Perf}\, Y$,

Note that there are no assumptions on the singularities of X and Y, and the hypotheses that X and Y be projective can be relaxed somewhat. As indicated before, the strategy of proof is to construct an isomorphism from the functor $F : \mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$, and from this construction, we can also determine the autoequivalence group of the derived category.

Corollary 2.1. *Let X be a curve of arithmetic genus different then one. Then*

$$\mathrm{Aut}(\mathrm{D}^b(X)) = \mathrm{Aut}(\mathrm{Perf}\, X) \cong \mathrm{Aut}\, X \ltimes (\mathrm{Pic}\, X \oplus \mathbb{Z}).$$

3. Indecomposability for some Cohen-Macaulay varieties

As in many areas of math it is often useful to decompose the objects you are studying into smaller, more manageable chunks. The derived category is no exception, and in this setting, the right notion of a decomposition is known as a semiorthogonal decomposition. To be precise, they are defined as follows.

Definition 3.1. *Let $\mathcal{T}$ be a triangulated category. A semiorthogonal decomposition of $\mathcal{T}$ is a collection of full admissible triangulated subcategories $\mathcal{A}_1, ..., \mathcal{A}_n$ of $\mathcal{T}$, called the components, such that*

(1) $\mathrm{Hom}_{\mathcal{T}}(A_j, A_i) = 0$ *for all $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$, $j > i$,*
(2) *The smallest triangulated subcategory containing all of the $\mathcal{A}_i$, $i = 1, ..., n$ is $\mathcal{T}$.*

The existence of semiorthogonal decompositions often has drastic implications for the geometry of the variety in question, typically related to the rationality of the variety. For an example from noncommutative geometry, consider when $\mathcal{A}_i \cong \mathrm{D}^b(\mathrm{Vect})$, the derived category of vector spaces, for all i. Under some additional assumptions, results of Bondal [Bon89] show that $\mathrm{D}^b(X)$ is then equivalent to the derived category of representations of a specific directed quiver, which is explicitly constructed from the categories $\{\mathcal{A}_i\}$. On the more geometric side, it is conjectured for various X that if all of the components $\mathcal{A}_i$ are equivalent to $\mathrm{D}^b(Y_i)$, where Y_i is a variety with $\dim Y_i \leq \dim X - 2$, then X is rational [BB12]. In particular, one can see rather directly that the derived category of a cubic fourfold $W \subset \mathbb{P}^5$ admits a decomposition

$$\mathrm{D}^b(W) \cong \langle \mathcal{A}, \mathcal{O}_W, \mathcal{O}_W(1), \mathcal{O}_W(2) \rangle.$$

It is conjectured that $\mathcal{A}$ is equivalent to the derived category of a K3 surface precisely when W is rational.

While useful, semiorthogonal decompositions can be hard to produce without inspiration, but there are several cases which are well-understood. Of particular interest are the categories which are indecomposable, in the sense that they admit no nontrivial semiorthogonal decompositions. For example, the derived category of a Calabi-Yau variety or of a smooth projective curve (of positive genus). More generally, it was shown in [KO15] that if the base locus[4] of the canonical bundle is small enough, then the derived category is indecomposable.

In forthcoming work [Spe20a], I was able to generalize this result of Kawatani and Okawa to the case of Cohen-Macaulay varieties. Specifically, the following result is the main goal of the paper.

Theorem 3.1. *Suppose that X is a Cohen-Macaulay projective variety such that every connected component of $\mathrm{Bs}\,|\omega_X|$ is contained in an open subset on which ω_X is trivial. Then $\mathrm{Perf}\, X$ is indecomposable. If $\mathrm{Bs}\,|\omega_X|$ is empty, then $\mathrm{D}^b(X)$ is also indecomposable.*

In this forthcoming work, I also hope to be able to show that all singular curves have an indecomposable category of perfect complexes, generalizing another result of Okawa which showed that smooth curves have indecomposable derived categories.

4. Future work

The above two projects that I have completed have supplied many further directions for investigation in the future. Although I have several questions in mind, the most immediately actionable in my opinion are the following.

Project 1: Reconstruction of singular varieties and derived equivalences. In light of Theorem 2.1, it is a natural question to ask if a similar result holds for varieties of higher dimension with possibly worse singularities. Considering only the former question for the moment, the proof contained in [Spe20b] will generalize to arbitrary dimensions so long as the Cohen-Macaulay singularities are isolated, and there is a suitable notion of amplitude for the dualizing sheaf ω_X. There are a few competing notions of amplitude for a general sheaf, several among these seem to plausibly work for a generalization.

In addition, allowing worse singularities seems to also be possible. The method of proof of Theorem 2.1 uses objects which are only defined for Cohen-Macaulay varieties. However by the work of [SdS09], it seems there are replacements which satisfy roughly the same properties. Thus it is plausible that an extension (granting the previous paragraph) to the case of arbitrary singularities is also possible.

[4] The base locus of a vector bundle is the common vanishing set of its global sections.

On the other hand, the question of when two nonisomorphic varieties can be derived equivalent is also a very interesting question. Such varieties are comparatively rare, needing ω_X to be neither ample nor anti-ample, but examples can be found among Calabi-Yau varieties, that is, varieties with ω_X isomorphic to the trivial line bundle. When the two varieties are smooth, many examples are known, but when considering singular varieties (for a suitable generalization of the definition of Calabi-Yau) the only known examples are contained in [MRV19], which provides examples of derived equivalent but nonisomorphic "singular abelian varieties". In dimension two the only other class of Calabi-Yau varieties are K3 surfaces, and in particular providing examples of derived equivalent singular K3 surfaces would be interesting for many reasons.

Project 2: The singularity category of curves. In my two projects detailed in the previous sections, we dealt with two main categories, $\mathrm{D}^b(X)$, and the subcategory of perfect objects, $\operatorname{Perf} X$. It is a well-known fact that X is regular if and only if these two categories agree, but in the singular case, it is interesting to understand the discrepancy between the two.

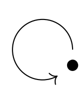
To this end, one can define the singularity category, given as the quotient $\mathcal{S}(X) = \mathrm{D}^b(X)/\operatorname{Perf} X$. Despite the fact that this category is closely connected to singularities (which a priori could be very complicated), it enjoys a number of useful properties [Orl06]. For example, if $U \subset X$ is an open subset which contains the singular locus, then $\mathcal{S}(U) \cong \mathcal{S}(X)$. If X is Gorenstein, then one can actually give a very explicit description of the singularity category. For worse singularities however, very little is known.

My proposed project is to explore the struture of the singularity category when X possesses non-Gorenstein singularities. Similar to previous projects, it makes sense to first investigate the structure for Cohen-Macaulay singularities. In particular, I propose to explore the structure of $\mathcal{S}(X)$ when X is the coordinate axes in affine 3-space, which is one of the simplest examples of a non-Gorenstein singularity of curves. From this I hope to extract some clues as to what the structure of the singularity category might be in more general situations.

5. Undergraduate research

While my work is very abstract, it supplies many interesting connections to more concrete subjects in which undergraduates can contribute in a meaningful way with only a course in algebra as a prerequisite. As mentioned earlier, when the derived category of a variety X admits a decomposition into pieces equivalent to the derived category of vector spaces, then the homological algebra encoded in $\mathrm{D}^b(X)$ can be effectively studied by instead working with representations of a directed quiver.

The study of quiver representations is essentially a generalization of linear algebra. For example, the problem of understanding the (isomorphism classes of) representations of the quiver



is the same as classifying endomorphisms of a vector space under conjugation, i.e., Jordan canonical form. For this reason quivers are especially approachable and computational, and the study of quivers is also full of interesting and explicit connections to geometry, but also to more surprising topics such as topological data analysis. In particular, there are many open questions that could be attempted by an undergraduate with proper supervision, requiring only a good understanding of linear algebra and some modest abstract algebra.

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