Tangent Spaces : First-Order Deformation Theory

Denote by $\mathbb{C}[\varepsilon]$ the ring of dual numbers, and $\Sigma = \operatorname{Spec} \mathbb{C}[\varepsilon]$, with s_0 its closed point. Recall that the tangent space to $H = \mathrm{Hilb}_r^{P(t)}$ at a point h corresponding to $X_0 \subset \mathbb{P}^n$ is exactly the set of pointed morphisms of pointed schemes $\operatorname{Hom}((\Sigma, s_0), (H, h))$. Since H represents the Hilbert functor, we can see this as the collection of all flat families $X \longrightarrow \Sigma$ with $X_{s_0} = X_0$.

So the tangent space at h can be studied by looking at first-order embedded deformations of X_0 in $\mathbb{P}^n$. The following lemma will prove to be important:

Lemma: Let $\varphi : A \to B$ be a homomorphism of noetherian commutative rings, with B flat over A. Assume either A is a local artinian ring or that A and B are both local rings and φ is a local homomorphism. Let J be an ideal in B, and set $C = B/J$. Let k be the quotient of A by its maximal ideal m_A, and set $B_k = B \otimes_A k$, $C_k = C \otimes_A k$. Then the following are equivalent

1) C is flat over A;
2) Every exact sequence $B_k^l \to B_k^h \to B_k \to C_k \to 0$ is the reduction modulo m_A of an exact sequence $B^l \to B^h \to B \to C \to 0$;
3) There are generators $F_1, \dots, F_h$ of J such that, denoting by f_i the image of F_i in B_k, every relation among the f_i extends to a relation among the F_i.

One should keep the special case of $A = \mathbb{C}[\varepsilon]$ and $B = R[\varepsilon] = R \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]$ in mind, being the setup of affine deformations. We omit the proof since it's purely algebraic, rather we use it to prove the following:

Lemma: Let R be a commutative noetherian $\mathbb{C}$-algebra, and let $I \subset R$ be an ideal. The first-order embedded deformations of $X_0 = \operatorname{Spec}(R/I)$ within $Y = \operatorname{Spec} R$ are in one-to-one correspondence with $\operatorname{Hom}_{R/I}(I/I^2, R/I) = \operatorname{Hom}_R(I, R/I)$.

Proof: Given $g \in R$, we write $[g]$ for its class modulo I. We must classify ideals $J \subset R[\varepsilon]$ such that $R[\varepsilon]/J$ is flat over $\mathbb{C}[\varepsilon]$ and $J/((\varepsilon) \cap J) = I$. Let such an ideal J be given, and given $i \in I$, choose a $j \in J$ whose reduction modulo (ε) is i. We can then write $j = i - \varepsilon h$, where h depends on j and is uniquely determined by i. Indeed if $i = 0$, then εh belongs to both J and (ε). The above lemma can be used to show that the flatness of $R[\varepsilon]/J$ implies $J \cap (\varepsilon) = \varepsilon J = \varepsilon I$, so $h \in I$. Now given $\alpha : I \longrightarrow R/I$, choose generators $f_1, \dots, f_n$ for I, write $\alpha(f_i) = [g_i]$ where $g_i \in R$, and $F_i = f_i - \varepsilon g_i$ and $J = (F_1, \dots, F_n)$. Clearly $J/J \cap (\varepsilon) = I$. Now we only need to show $R[\varepsilon]/J$ is flat over $\mathbb{C}[\varepsilon]$.

By the lemma, its enough to show that any relation among the f_i is the reduction modulo (ε) of a relation among the F_i. Now let

$$\sum_i a_i f_i = 0$$

be a relation. Notice that

$$\sum a_i [g_i] = \alpha\left(\sum a_i f_i\right) = 0,$$

meaning $\sum a_i g_i \in I$, so $\sum a_i g_i = \sum b_i f_i$ for some $b_i \in R$. Thus we see

$$\sum (a_i + \varepsilon b_i) F_i = \sum a_i f_i + \varepsilon \left(\sum b_i f_i - \sum a_i g_i \right) = 0$$

is a relation among the F_i which extends $\sum a_i f_i = 0$. So to each ideal $J \subset R[\varepsilon]$ extending I and $R[\varepsilon]/J$ is flat over $\mathbb{C}[\varepsilon]$, a homomorphism of R-modules from I to R/I, and conversely. ∎

Now let X be a closed subscheme of a fixed scheme Y, and let $\mathcal{J}$ be the ideal of X in Y. The sheaf $\mathcal{J}/\mathcal{J}^2|_X$ is called the conormal sheaf of X in Y and denoted $C_{X/Y}$. It's dual $\mathcal{H}om_{\mathcal{O}_X}(C_{X/Y}, \mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{J}, \mathcal{O}_X)$ is the normal sheaf of X in Y, denoted $N_{X/Y}$. From the above, we immediately arrive at the following:

Prop: Let X be a closed subscheme of Y, and $\mathcal{J}$ the ideal sheaf of X in Y. Then the first-order embedded deformations of X in Y are in one-to-one correspondence with $H^0(X, N_{X/Y})$.

Now let $Y \times B \supset \mathcal{X} \longrightarrow B$ be a flat family of subschemes of Y parametrized by B. Set $X = X_{b_0}$ for some closed point $b_0 \in B$, and v a tangent vector to B at b_0. Then we can interpret v as a map $\operatorname{Spec} \mathbb{C}[\varepsilon] \longrightarrow (B, b_0)$. Pulling back $\mathcal{X}$ via v gives a first order deformation of X in Y, i.e. an element of $H^0(X, N_{X/Y})$. Thus we get a map

$$T_{b_0} B \longrightarrow H^0(X, N_{X/Y}),$$

which Kodaira and Spencer called the characteristic map. It turns out to be $\mathbb{C}$-linear.

Now consider again $\mathrm{Hilb}_r^{p(t)}$, with h a closed point representing X in $\mathbb{P}^r$. Then we conclude:

$$T_h(\mathrm{Hilb}_r^{p(t)}) = H^0(X, N_{X/\mathbb{P}^r}).$$

Thus we see $h^0(X, N_{X/\mathbb{P}^r})$ is an upper bound for the dimension of $\mathrm{Hilb}_r^{p(t)}$ at h. One can show (but we will not) that the dimension of every irreducible component at h has dimension at least $h^0(X, N_{X/\mathbb{P}^r}) - h^1(X, N_{X/\mathbb{P}^r})$.

We now give an important example:

Example: Consider a smooth complete nondegenerate degree d, genus g curve $C \subset \mathbb{P}^r$. Then C gives a point $[C]$ in $\mathrm{Hilb}_r^{p(t)}$, with $p(t) = dt + 1 - g$ (Riemann-Roch). From the above $\dim_{[C]} H \le h^0(C, N_{C/\mathbb{P}^r})$, and we use the standard exact sequence for the normal sheaf:

$$0 \longrightarrow T_C \longrightarrow T_{\mathbb{P}^r} \otimes \mathcal{O}_C \longrightarrow N_{C/\mathbb{P}^r} \longrightarrow 0.$$

Hence $\deg N_{C/\mathbb{P}^r} = \deg(T_{\mathbb{P}^r} \otimes \mathcal{O}_C) - \deg T_C = (r+1)d - (2-2g) = (r+1)d + 2g - 2$.
Now Riemann-Roch for higher rank bundles gives:

$$\begin{aligned} \chi(N_{C/\mathbb{P}^r}) &= h^0(C, N_{C/\mathbb{P}^r}) - h^1(C, N_{C/\mathbb{P}^r}) \\ &= \deg(N_{C/\mathbb{P}^r}) - (\operatorname{rank} N_{C/\mathbb{P}^r})(g-1) \qquad \text{(Pf: Hirzebruch-Riemann-Roch).} \\ &= (r+1)d - (r-3)(g-1). \end{aligned}$$