# Basic Definitions & Properties (Based on Notes by Brion)

Fix our base field to be $\mathbb{C}$. We consider algebraic varieties (possibly not irreducible) over $\mathbb{C}$. We will just call these varieties, and unless otherwise stated they are equipped with the Zariski topology.

**Def**: An algebraic group is a variety $G$, equipped with the structure of a group, such that:
$$\left\{\begin{array}{ll} \mu: G\times G \to G, & (g,h)\mapsto gh \\ i: G \to G, & g \mapsto g^{-1} \end{array}\right\}$$
are morphisms.

**Examples**:
1) Any finite group.
2) $GL_n(\mathbb{C})$, the general linear group of $n\times n$ matrices. Indeed if we regard $M_n$ as an affine variety with the matrix entries as indeterminates, we see that $GL_n(\mathbb{C}) = D(\Delta) \subset M_n$, where $\Delta$ is the determinant polynomial. Restricting the structure sheaf, we get that $GL_n(\mathbb{C})$ is an affine variety.
3) Any closed subset of $GL_n(\mathbb{C})$, for example, $SL_n(\mathbb{C}) = \{x \in GL_n(\mathbb{C}) \mid \Delta(x) = 1\}$.
4) The multiplicative group $\mathbb{C}^*$ and additive group $\mathbb{C}$. Indeed $\mathbb{C}^* \cong GL_1(\mathbb{C})$. More generally, let $T_n \subset GL_n(\mathbb{C})$ be the subgroup of diagonal matrices, we can see fairly easily that $T_n \cong (\mathbb{C}^*)^n$, and is called an algebraic torus.
5) Every smooth curve of degree 3 in $\mathbb{P}^2$ has the structure of an algebraic group. These are called elliptic curves. See [Hartshorne] or [Milne]'s book.

**Lemma**: Any algebraic group $G$ is smooth, its cosets are the translates $gG^0$ of the component of the identity, $G^0$ is closed, and $G/G^0$ is finite.
**Proof**: Only the first is not clear. Choose a nonsingular point $h \in G$. This can be done as $\mathrm{Sing}(G)$ has codimension at least one. Then since $G$ is a group, and multiplication is a morphism, $gh$ is nonsingular $\forall g \in G$. Since this action is transitive, we are done. ∎

**Def**: A $G$-variety $X$ is a variety with an action by an algebraic group $G$
$$\alpha: G\times X \to X, \quad (g,x) \mapsto \alpha(g,x) = g\cdot x$$
which is a morphism of varieties.

Such an action gives a natural action on regular functions on $X$ via $(g\cdot f)(x) = f(g^{-1}\cdot x)$.

**Lemma**: If $X$ is an affine $G$-variety, then the complex vector space $\Gamma(X)$ is a union of finite dimensional $G$-stable subspaces.
**Proof**: The action map $\alpha: G\times X \to X$ induces a morphism of coordinate rings:
$$\alpha^*: \Gamma(X) \longrightarrow \Gamma(G\times X),$$
taking $f \mapsto ((g,x) \mapsto f(g\cdot x))$, called the coaction. Since $\Gamma(G\times X) = \Gamma(G)\otimes_{\mathbb{C}} \Gamma(X)$, we can write
$$f(g\cdot x) = \sum_{i=1}^{n} \varphi_i(g)\,\psi_i(x).$$

Then:
$$g\cdot f = \sum_{i=1}^{n} \varphi_i(g^{-1})\,\psi_i,$$

and hence the translates $g \cdot f$ span a finite dimensional subspace $V \subset \Gamma(X)$, which is clearly $G$-stable. ∎

This motivates the notion of a rational $G$-module, namely a vector space with an action by $G$ such that every element is contained in a finite dimensional $G$-stable subspace. One example is of course affine coordinate rings as above. Note that finite dimensional $G$-modules are in one-to-one correspondence with homomorphisms of algebraic groups $G \to GL_n$ for some $n$.

<u>Example</u>: Let $G = \mathbb{C}^*$. Then

$$\Gamma(G) = \mathbb{C}[t, t^{-1}] = \bigoplus_{n=-\infty}^{\infty} \mathbb{C}t^n.$$

https://mathoverflow.net/questions/104756/action-of-k-on-a-variety-

Given a $G$-variety $X$, any $f \in \Gamma(X)$ satisfies

$$(t \cdot f)(x) = f(t^{-1} x) = \sum_{n=-\infty}^{\infty} t^n f_n(x)$$

where the $f_n$ are uniquely determined by $f$. For example if $f = x^2 + y$, then $(t \cdot f)(x,y) = t^{-2}x^2 + t^{-1}y = \sum_{n=-2}^{-1} t^n f_n$, where $f_{-2} = x^2$ and $f_{-1} = y$. Setting $t=1$ gives $f = \sum_n f_n$.

From the above then, we see that each term obeys $t \cdot f_n = t^n f_n$. If we now set $\Gamma(X)_n$ to be the span of each $f_n$ $\forall f \in \Gamma(X)$, then we can check that $t \cdot f_n f_m = t^{n+m} f_n f_m$, so the action gives a $\mathbb{Z}$-grading of finite dimensional vector spaces:

$$\Gamma(X) = \bigoplus_{n=-\infty}^{\infty} \Gamma(X)_n,$$

moreover, by reversing the above, any $\mathbb{Z}$-graded ring with no non zero nilpotents give a $\mathbb{C}^*$-variety.

Even more generally, the coordinate ring of an algebraic torus $(\mathbb{C}^*)^n$ is $\mathbb{C}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$. and the action of $T_n$ on affine varieties gives a $\mathbb{Z}^n$-graded structure.

<u>Def</u>: Given two $G$-varieties $X, Y$, a morphism $f: X \to Y$ is called equivariant if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$, $x \in X$. We also say $f$ is a $G$-morphism.

<u>Def</u>: Given a $G$-variety $X$ and a point $x \in X$, the orbit $G \cdot x \subset X$ is the set of all $g \cdot x$. The isotropy group (also called the stabilizer) $G_x \subset G$ is the subgroup of elements $g \in G$ with $g \cdot x = x$.

The orbit $G \cdot x$ is a locally closed smooth subvariety of $X$, and every component has dimension $\dim G - \dim G_x$. Moreover the closure $\overline{G \cdot x}$ is the union of $G \cdot x$ and orbits of strictly smaller dimension. Any orbit of minimal dimension is closed.

Examples:

1) Consider $\mathbb{C}^*$ acting on affine $n$-space via $t\cdot(x_1,\dots,x_n) = (tx_1,\dots,tx_n)$. Then the origin is the unique closed orbit, and the orbit closures are exactly the lines through $0$.
2) Let $\mathbb{C}^*$ act on $\mathbb{C}^2$ via $t\cdot(x,y) = (tx, t^{-1}y)$. Then the closed orbits are the origin and points on the hyperbolae $xy = c$, $c \neq 0$. The other orbit closures are the coordinate axes.

Applying what we know to algebraic group homomorphisms, if $f: G \to H$ is such a map, then $\mathrm{Im}(f) \subset H$ is a closed subgroup. If $\mathrm{Ker}(f) = 0$, then $f$ is a closed immersion. One can use this to show that any affine algebraic group is linear.

We have the following important result, due to Chevalley.

<u>Thm:</u> Let $G$ be a linear algebraic group, and $H \subset G$ a closed subgroup. Then there exists a finite-dimensional $G$-module $V$ and a line $\ell \subset V$ such that $G_\ell = H$.

<u>Proof</u>: Consider $G$ acting on itself by left multiplication. Then the stabilizer of the closed subvariety $H$ is $H$ itself. Then considering the associated action on $\Gamma(G)$, $H$ is the stabilizer of $I(H) \subset \Gamma(G)$. We can choose a vector subspace $W \subset I(H)$ which generates the ideal, and since $I(H)$ is $H$-stable, we can assume $W$ is $H$-stable. Then $W$ is contained in a finite-dimensional $G$-submodule $V \subset \Gamma(G)$. Since $H$ is the stabilizer of $W$, and so $H$ is also the stabilizer of the line $\Lambda^n W$ of the $G$-module $\Lambda^n V$, $n = \dim W$. ∎

The above result can be rephrased in a more geometric context. Namely given a finite-dimensional $G$-module $V$, any closed subgroup of $G$ is the stabilizer of a point in $\mathbb{P}(V)$. This is the starting point for the construction of quotients of linear algebraic groups by closed subgroups.

<u>Thm:</u> Let $G$ be a linear algebraic group, and $H$ a closed subgroup. Then the coset space $G/H$ has a unique structure of a $G$-variety which satisfies:

1) The quotient $\pi: G \to G/H$ is a morphism.
2) A subset $U \subset G/H$ is open if and only if $\pi^{-1}(U)$ is open.
3) For any open $U \subset G/H$, the comorphism $\pi^*$ yields an isomorphism $\Gamma(U, \mathcal{O}_G) \cong \Gamma(\pi^{-1}(U), \mathcal{O}_G)^H$

Moreover, $G/H$ is smooth and quasi-projective.

We won't prove this here, as it involves some notions of smoothness from scheme theory, but the quotient map is easy to construct. We may choose a $G$-module $V$ and a point $x \in \mathbb{P}(V)$ such that $H = G_x$. Then the map $\pi: G \to X$, $g \mapsto g\cdot x$, where $X = G\cdot x$ is surjective, and has fiber $\pi^{-1}(g\cdot x) = gH$.

Note that the third statement is equivalent to the statement that the natural map $\mathcal{O}_X \to (\pi_* \mathcal{O}_G)^H$ is an isomorphism.

We can now make the following definition.

**Def:** A variety $X$ is homogeneous if it is equipped with a transitive action of an algebraic group $G$. A homogeneous space is a pair $(X, x)$, where $X$ is a homogeneous variety and $x \in X$ a point, called the base point.

By the theorem above, the homogeneous spaces $(X, x)$ under a linear algebraic group are exactly the quotient spaces $G/H$, where $H = G_x$, with base point the coset $H$.

We now move toward discussing quotients of $G$-varieties.

**Def:** Given an algebraic group $G$ and a $G$-variety $X$, a geometric quotient of $X$ by $G$ consists of a pair $(Y, \pi)$, $\pi : X \to Y$ satisfying:
1) $\pi$ is surjective, and its fibers are the orbits of $G$ in $X$.
2) $U \subset Y$ is open if and only if $\pi^{-1}(U)$ is open.
3) The natural map $\mathcal{O}_Y \to (\pi_* \mathcal{O}_X)^G$.

As before, we are able to identify $Y$ with the orbit space $X/G$, equipped with the quotient topology, moreover the structure of a variety on $Y$ is uniquely determined by $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$. We know already that geometric quotients of algebraic groups by closed subgroups exist, but in general they may not.

**Examples:**
1) Consider $\mathbb{C}^*$ acting on $\mathbb{C}^n$ by scalar multiplication. Since $0$ lies in every orbit closure, there cannot be a geometric quotient. However the space $\mathbb{C}^n \setminus \{0\}$ does, namely $\mathbb{P}^{n-1}$.
2) Let $G = \mathbb{C}^*$ act on $\mathbb{C}^2$ via $t \cdot (x, y) = (tx, t^{-1}y)$. Then $X = \mathbb{C}^2 \setminus \{0\}$ is a $G$-stable open subset where all orbits are closed, but admits no geometric quotient. Indeed $K(X)^G = K(xy)$, and so the orbits $x = 0$ and $y = 0$ are not separated by $G$-invariant rational functions.

A theorem of Rosenlicht that any irreducible $G$-variety $X$ contains a non-empty open $G$-stable subset $X_0$ which admits a geometric quotient $Y_0 = X_0/G$. However there may not be an obvious choice for $X_0$. We may also try weakening the notion of a quotient, maybe just parametrizing the closed orbits.

**Def:** A linear algebraic group $G$ is reductive if it does not contain any closed normal unipotent subgroup.

If the reader does not know the definition of unipotent, let $U_n \subset GL_n$ be the subgroup of upper triangular matrices with diagonal entries 1. The closed subgroups of $U_n$ are called unipotent. They are all nilpotent.

There is another characterization of unipotent that is more appropriate for our use, based on the following:

**Def:** Let $G$ be an algebraic group and $V$ a rational $G$-module. Then $V$ is simple (also called irreducible) if it has no proper nonzero $G$-submodule. $V$ is semi-simple (also called completely reducible) if it satisfies one of the following equivalent conditions:
1) $V$ is isomorphic to a direct sum of simple $G$-modules.
2) Any submodule $W \subset V$ admits a $G$-stable complement, i.e., a $G$-submodule $W'$ such that $V = W \oplus W'$.

The relevant fact for our purposes is every simple $G$-module is trivial, i.e., isomorphic to $\mathbb{C}$ where $G$ acts trivially. The proof of this is not too hard of a fact, it only requires some elementary representation theory. In particular, any non-zero module under a unipotent group action contains non-zero fixed points.

**Thm:** The following assertions are equivalent for a linear algebraic group $G$:
1) $G$ is reductive;
2) $G$ contains no closed normal subgroup isomorphic to the additive group $\mathbb{C}^n$ for some $n \geq 1$;
3) $G$ (viewed as a Lie group w/ the $\mathbb{C}$-topology) has a compact subgroup $K$ which is dense in the Zariski topology;
4) Every finite-dimensional $G$-module is semi-simple;
5) Every $G$-module is semi-simple.

We now come to a key result.

**Thm:** Let $G$ be a reductive algebraic group and $X$ an affine $G$-variety. Then:
1) The subalgebra $\Gamma(X)^G \subset \Gamma(X)$ is finitely generated.
2) Let $f_1, \dots, f_n$ be generators of $\Gamma(X)^G$. Then the image of the morphism
$$X \to \mathbb{C}^n, \quad x \mapsto (f_1(x), \dots, f_n(x)),$$
is closed and independent of the choice of $f_1, \dots, f_n$.
3) Denote by $\pi = \pi_X : X \to X/\!/G$ the surjective morphism above. Then every $G$-invariant morphism $f: X \to Y$, where $Y$ is an affine variety, factors through a unique morphism $\varphi: X/\!/G \to Y$.
4) For any closed $G$-stable subset $Y \subset X$, the induced morphism $Y/\!/G \to X/\!/G$ is a closed immersion, in particular, $\pi_X|_Y = \pi_Y$. Moreover, given another closed $G$-stable subset $Y'$, we have $\pi_X(Y \cap Y') = \pi_X(Y) \cap \pi_X(Y')$.
5) Each fiber of $\pi_X$ contains a unique closed $G$-orbit.
6) If $X$ is irreducible, then so is $X/\!/G$. If $X$ is normal, then so is $X/\!/G$.

We wont prove this, but the proof isn't too difficult. The key invention is the so-called Reynolds operator. For a $G$-module $V$, we have the $G$-submodule of invariants $V^G$, and the $G$-stable complement $V_G$. Then $R_V$ is defined as the composition:

$$R_V : V \xrightarrow{\sim} V^G \oplus V_G \xrightarrow{\pi} V^G$$

Note that $\pi$ is uniquely defined by the universal property (3); it is called a categorical quotient. In addition, $X/\!/G$ may be viewed as the space of closed orbits by (5), and so one might suspect that this can be related to a geometric quotient. This instinct is correct.

**Def:** Let $G$ be an algebraic group, and $X$ an affine $G$-variety. A point $x \in X$ is stable if $G \cdot x$ is closed in $X$ and the isotropy group $G_x$ is finite. The (possibly empty) set of stable points is denoted $X^s$

**Prop:** With the preceeding notation and assumptions, $\pi(X^s)$ is open in $X/\!/G$, we have $X^s = \pi^{-1}\pi(X^s)$ (in particular, $X^s$ is a $G$-stable open subset of $X$), and the restriction $\pi^s : X^s \to \pi(X^s)$ is a geometric quotient.

**Proof:** Let $x \in X^s$, and consider $Y \subset X$ consisting of points with infinite isotropy subgroup. Note this is the same as $\dim G_y > 0$. Then $Y$ is closed, $G$-stable and disjoint from $G \cdot x$ (as $G_{gx} = g G_x g^{-1}$, which is also finite). Thus, there is $f \in \Gamma(X)^G$ such that $f(x) \neq 0$ and $f|_Y$ is identically zero. Then the open subset $X_f$ is a subset of $X$ containing $x$, and satisfies $X_f = \pi^{-1}\pi(X_f)$ (as $f \in \Gamma(X)^G$).