You Can't Always Get What You Want....

We have seen so far that the Hilbert scheme $\mathcal{X} \xrightarrow{\pi} \mathrm{Hilb}_r^{P(t)}$ enjoyed a very nice universal property: for any other flat family $X \xrightarrow{p} H$ of subschemes of $\mathbb{P}^r$ with constant Hilbert polynomial $p(t)$, there is a unique map $\alpha: H \to \mathrm{Hilb}_r^{P(t)}$ making X the pullback:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ H \arrow[r] & \mathrm{Hilb}_r^{P(t)} \end{tikzcd}$$

and this property prompts the name of "universal family" for $\mathcal{X}$. More generally, any scheme which may be a candidate moduli space, a fine moduli space, if it has such a universal family with a universal property similar to the above.

However (as we will give an example of), such spaces are rare. So if fine moduli are "too much" to ask for, what could we want instead? One answer is to extend our category to hopefully get fine moduli, and this leads one to algebraic spaces and stacks (we will return to this). The other is to ask for less. First we give a formal definition.

Def: Fix some collection of objects (schemes, sheaves, etc.) and a notion of a family over a scheme B. Set $S(B)$ to be the set of all families of such objects over B, and also fix some (possibly trivial) equivalence relation $\sim$ on $S(B)$. We then define the moduli functor $F(B) = S(B)/\sim$. If F is representable by a scheme $\mathcal{M}$, we say $\mathcal{M}$ is a fine moduli space for this moduli problem.

So one compromise to the nonexistence is to just ask for a natural transformation from F to $\mathrm{Mor}(-,\mathcal{M})$ (instead of the isomorphism required by representability). However this leads to serious issues. We still have, for each $X \to B$, a morphism $B \to \mathcal{M}$, and compatibility with base change, but $\mathcal{M}$ would not be unique! Given any morphism $\pi: \mathcal{M} \to \mathcal{M}'$, $\mathcal{M}'$ is another candidate! We could take $\mathcal{M}' = \mathrm{Spec}\, \mathbb{C}$, and then we would be recording nothing of interest.

We can fix this by requiring complex points correspond bijectively to objects. This still isn't quite enough to fix the scheme structure as we can compose with a map which is bijective on complex points (take cuspidal curve $y^2z = x^3$ and $\mathbb{P}^1$). So we say $\mathcal{M}$ is universal with respect to the existence of the natural transformation.

Def: A scheme $\mathcal{M}$ and a natural transformation $\Psi_{\mathcal{M}}: F \to \mathrm{Mor}(-,\mathcal{M})$ are a coarse moduli space for F if:

1) The map $\Psi_{\mathcal{M}}: F(\mathrm{Spec}\,\mathbb{C}) \longrightarrow \mathcal{M}(\mathbb{C}) = \mathrm{Mor}(\mathrm{Spec}\,\mathbb{C}, \mathcal{M})$ is a set bijection.
2) Given another scheme $\mathcal{M}'$ and natural transformation $\Psi_{\mathcal{M}'}: F \longrightarrow \mathrm{Mor}(-,\mathcal{M}')$, there is a unique map $g: \mathcal{M} \to \mathcal{M}'$ inducing $G: \mathrm{Mor}(-,\mathcal{M}) \to \mathrm{Mor}(-,\mathcal{M}')$ with $\Psi_{\mathcal{M}'} = G \circ \Psi_{\mathcal{M}}$.

Example: Elliptic Curves over $\mathbb{C}$.

An elliptic curve is a genus one smooth curve with a choice of marked point p_0. Riemann-Roch tells you that $|2p_0| \cong \mathbb{P}^1$, so it is actually hyperelliptic. Counting the ramification points, we can see the hyperelliptic map is ramified at 4 points, and an automorphism sends these 4 to $0, 1, \infty, \lambda$ for some $\lambda \in \mathbb{C}$. The j-function of an elliptic curve is a rational function of λ.

λ does not determine the curve uniquely, as $S_3 \curvearrowright \{0,1,\lambda\}$ by permuting and applying an automorphism to get $\{0,1,\lambda'\}$ for $\lambda' \neq \lambda$. However its a well known fact (Hartshorne Theorem IV.4.1) that j is invariant under this action, and j is a complete invariant of the elliptic curve.

More specifically, given some $\tau \in \mathbb{H}$ (upper-half plane), we have a lattice $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\{\tau\}$, and $\mathbb{C}/\Lambda_\tau = E_\tau$ is an elliptic curve (in the $\mathbb{C}$-topology). We can embed E_τ in $\mathbb{P}^2$ with equation $(z=1)$ $y^2 = 4x^3 + g_2(\tau)x + g_3(\tau)$ using the Weierstrauss $\wp$-function. A way of expressing the j-function is then

$$j(\tau) = \frac{1728\, g_2(\tau)^3}{g_2(\tau)^3 - 27\, g_3(\tau)^2}.$$

Now $SL_2\mathbb{Z} \curvearrowright \mathbb{H}$ in the usual way (by fractional linear transformations), and it also happens that j is $SL_2\mathbb{Z}$ invariant. Now $SL_2\mathbb{Z} \ltimes \mathbb{Z}^2 \curvearrowright \mathbb{H} \times \mathbb{C}^2$ by:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (m_1, m_2)\right) \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + m_1 + m_2\tau}{c\tau + d}\right).$$

induced by projection.

Taking the total quotient, we get a variety $\mathcal{E}$ fibered over $SL_2\mathbb{Z}\backslash\mathbb{H} = Y$. Since $E_\tau \cong E_{\tau'}$ iff τ is in the $SL_2\mathbb{Z}$ orbit of τ', one might (reasonably) guess that $M_{1,1}$ is Y, and $\mathcal{E}$ the universal elliptic curve. But this is spectacularly wrong.

The rough idea is that $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL_2\mathbb{Z}$, which acts trivially on $\mathbb{H}$, yet is a nontrivial involution on the fibers. So upto isomorphism, the fibers are actually rational!

On the contrary though this is the coarse moduli space of elliptic curves.

Three ways to build Moduli of Curves

To get a moduli space, we look at curves with extra structure so that we get a parameter space, then quotient by the equivalence relation identifying the structures.

There are several ways of doing this, one of which is related to Teichmüller theory, but we will go through a way more familiar to us. For now, fix $g \geq 2$. Then its an easy fact that $3K_X$ is very ample, and so we can embed all smooth proper curves of the same genus in the same $\mathbb{P}^n$ with the same degree. Hence we can look at the Hilbert scheme of such subschemes, $Hilb_n^{P(t)}$, and quotient by the action of $PGL(n+1, \mathbb{C})$ to get rid of the projective embedding. This is a candidate for M_g.

However we have no idea that such a quotient exists! So we pause to discuss some aspects of G.I.T.