Constructing the Hilbert Scheme I

We begin with a sketch of the general strategy. Fix a projective space $\mathbb{P}^r$. We want to parametrize closed subschemes of $\mathbb{P}^r$ with fixed numerical parameters, i.e. degree, dimension, etc. These quantities are naturally encoded in the Hilbert polynomial, which again we define as $P_X(t) = \chi(\mathcal{O}_X(t)) = \sum (-1)^i h^i(X, \mathcal{O}_X(t))$.

To parametrize these varieties, we need the notion of a (flat) family. Given some ambient scheme Y, a family of subschemes of Y parametrized by S is a subscheme $X \subset Y \times S$ viewed as fibered over S via the projection:

$$\begin{tikzcd} Y \times S \supset X \arrow[d, "f"] \\ S \end{tikzcd}$$

and we write X_s for $f^{-1}(s) = X \times_S \operatorname{Spec} k(s)$, $s \in S$. So what we are asking for is that the Hilbert polynomial $P_{X_s}(t)$ be independent of s. This is essentially the notion of flatness, which we now recall.

Def: A module M over a commutative ring R is flat if the endofunctor $(-) \otimes_R M$ is exact. Equivalently, $\operatorname{Tor}_1^R(M, N) = 0$ for any f.g. R-module N. A morphism of schemes $\varphi: X \to S$ is flat if for any $x \in X$, the natural map $\mathcal{O}_{S,f(x)} \longrightarrow \mathcal{O}_{X,x}$ makes $\mathcal{O}_{X,x}$ a flat $\mathcal{O}_{S,f(x)}$-module.

Indeed, we have the following:

Thm (Theorem III.9.9 Hartshorne): Let S be an integral noetherian scheme. Let $X \subseteq \mathbb{P}^n \times S$ be a closed subscheme. For each point $s \in S$, consider the Hilbert polynomial $P_s(t) \in \mathbb{Q}[t]$ of the fiber X_s, considered as a closed subscheme of $\mathbb{P}^n_{k(s)}$. Then X is flat over S iff $P_s(t)$ is independent of s.

We won't prove this, see Hartshorne for it. One we will prove later is actually a rather strong result:

Thm: Let $r \geq 0$ be an integer, and $p(t)$ a rational polynomial. Then there is an n_0, depending only on r and $p(t)$, such that for any subscheme $X \subset \mathbb{P}^r$ with Hilbert polynomial $p(t)$, and any $n \geq n_0$, the map

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \xrightarrow{\varphi_X^n} H^0(X, \mathcal{O}_X(n))$$

is surjective, and the schme X is completely determined by its kernel, and moreover,

$$q(n) = \dim \operatorname{Ker} \varphi_X^n = \binom{n+r}{r} - p(n).$$

Now if $\mathrm{Hilb}_r^{p(t)}$ is the set of subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(t)$, we get an injective set-map:

$$\begin{aligned} \mathrm{Hilb}_r^{p(t)} &\hookrightarrow Gr(q(n), H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n))) \\ X &\longmapsto [\operatorname{Ker} \varphi_X^n]. \end{aligned}$$

The strategy of the construction is to then give the image of this a scheme structure.

We will call the result above the "uniform-n_0 theorem."

We will actually get more. We will get a "universal family" in the sense that given another family $\mathbb{P}^r \times S \supset X \xrightarrow{f} S$ of closed subschemes with the same Hilbert polynomial, is the pullback of the universal family $\mathcal{X}$ over $\mathrm{Hilb}_r^{P(t)}$, via a unique $\alpha: S \to \mathrm{Hilb}_r^{P(t)}$:

$$\begin{array}{ccc} \mathbb{P}^r \times S \supset X & \longrightarrow & \mathcal{X} \subset \mathbb{P}^r \times \mathrm{Hilb}_r^{P(t)} \\ \big\downarrow f & & \big\downarrow \pi \\ S & \xrightarrow[\alpha]{} & \mathrm{Hilb}_r^{P(t)} \end{array}$$

Another way of saying this is that given the Hilbert functor Hilb: Schemes $\longrightarrow$ Sets,

$$\mathrm{Hilb}(S) = \left\{ \begin{array}{l} \text{families of closed subschemes of } \mathbb{P}^r \text{ w/ constant Hilbert polynomial } p(t), \\ \text{parametrized by } S \end{array} \right\}.$$

That $\mathrm{Hilb}_r^{P(t)}$ represents this functor.

We devote the rest of today for an example of a Hilbert scheme (before the construction). Next time we will prove the uniform-n_0 theorem and construct the Hilbert scheme.

<u>Example: Hypersurfaces</u>

Fix a projective space $\mathbb{P}^r$. If $X \subset \mathbb{P}^r$ is a hypersurface of degree d, we can compute it's Hilbert polynomial from taking cohomology of:

$$0 \to \mathcal{O}_{\mathbb{P}^r}(n-d) \longrightarrow \mathcal{O}_{\mathbb{P}^r}(n) \longrightarrow \mathcal{O}_X(n) \longrightarrow 0,$$

so $P_X(n) = \binom{n+r}{r} - \binom{n-d+r}{r} = \frac{d}{(r-1)!} t^{r-1} + \cdots$.

In fact more is true, any subscheme of $\mathbb{P}^r$ with this Hilbert polynomial is a hypersurface, checking this is elementary.

Set $N = \binom{d+r}{r} - 1$. If $x_0, \dots, x_r$ are homogeneous coords on $\mathbb{P}^r$, and a_I, $I = (i_0, \dots, i_r)$ a multi-index with $i_k \geq 0$, $\sum i_k = d$, are homogeneous coords on $\mathbb{P}^N$, we define the hypersurface $\mathcal{X} \subset \mathbb{P}^r \times \mathbb{P}^N$ defined by $\sum a_I x^I = 0$.

It is evident that $\mathcal{X}$ with $\pi: \mathcal{X} \to \mathbb{P}^N$ is $\mathrm{Hilb}_r^{P(t)}$ for hypersurfaces, $\mathcal{X}$ the universal family. Since the Hilbert polynomial is constant in the family, π is certainly flat, but we need to show the universal property. Let

$$\mathbb{P}^r \times S \supset X \xrightarrow{f} S$$

be a flat family of hypersurfaces of degree d. If $\mathcal{I}$ is the ideal sheaf of $X \subset \mathbb{P}^r \times S$, standard results on flatness and base change tells us that $f_* \mathcal{I}(d)$ is an invertible sheaf on S, and hence we can find an open cover $\{U_i\}$ and local generators σ_i on each U_i. Each σ_i may be thought of as a homogeneous polynomial $\sum b_{i,I} x^I$, with $b_{i,I}$ regular functions on U_i, and

$$f^{-1}(s) = V\left(\sum b_{i,I}(s) x^I\right) \subset \mathbb{P}^r, \quad s \in S.$$

Hence we get $\alpha_i : U_i \longrightarrow \mathbb{P}^N$ by $s \longmapsto [\cdots : b_{i,I}(s) : \cdots]$ which we can check glue to a global morphism $\alpha : S \to \mathbb{P}^N$, and our local description makes it clear that $X \xrightarrow{f} S$ is the pullback of $\mathcal{X} \xrightarrow{\pi} \mathbb{P}^N$. The uniqueness of α is skipped for now.

Example: Hilbert Schemes of Points

For a zero-dimensional subscheme Z of $\mathbb{P}^r$, we have seen its Hilbert polynomial is $d = \deg Z = \dim H^0(Z, \mathcal{O}_Z)$. Let H be the corresponding Hilbert scheme. Now consider the d-fold symmetric product of $\mathbb{P}^r$ minus the diagonal. This is a smooth variety S which is the parameter space of d distinct points of $\mathbb{P}^r$. By universality, S maps into H, and this is actually an open embedding. We will see that H is a compactification of S, but is not just the symmetric product in most cases, and exhibits many pathological properties.

Constructing the Hilbert Scheme II

We aim to now prove the uniform n_0-lemma, the heavy lifting being done by the following:

Lemma: Let r be a non-negative integer, and $g(t)$ a rational polynomial. Then there exists an integer n_0, such that, for any ideal sheaf $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}^r}$ with Hilbert polynomial $g(t)$ and any $n \geq n_0$,

1) $H^i(\mathbb{P}^r, \mathcal{J}(n)) = 0 \quad \forall i \geq 1$,
2) the natural map $H^0(\mathbb{P}^r, \mathcal{J}(n)) \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \longrightarrow H^0(\mathbb{P}^r, \mathcal{J}(n+1))$ is surjective.

Proof: We proceed by induction on r, the dimension of projective space, the case $r=0$ of course trivial. So assume $r>0$, and let X be the projective scheme determined by $\mathcal{J}$, and note that we may choose a hyperplane H not containing any connected components of X (including embedded ones). Set $\mathcal{F} = \mathcal{J} \otimes \mathcal{O}_H$, this can be interpreted as the ideal sheaf of $X \cap H$, regarded as a subscheme of H, and so we find that $\mathcal{F}$ is a sheaf of ideals in $\mathcal{O}_H$. Consider

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^r}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow \mathcal{O}_H \longrightarrow 0,$$

and tensor w/ $\mathcal{J}(m+1)$. We get the exact sequence:

$$0 \longrightarrow \mathcal{J}(m) \longrightarrow \mathcal{J}(m+1) \longrightarrow \mathcal{F}(m+1) \longrightarrow 0.$$

Thus $P_{\mathcal{F}}(t) = P_{\mathcal{J}}(t) - P_{\mathcal{J}}(t-1) = g(t) - g(t-1)$. Now by induction, since $H \cong \mathbb{P}^{r-1}$, $P_{\mathcal{F}}(t)$ depends only on $g(t)$, not on $\mathcal{J}$ or H. So there is an n_1 such that (1) + (2) hold for all $n \geq n_1$. In particular, this implies that $h^i(H, \mathcal{F}(n)) = 0$ for $i \geq 1$, and hence $H^i(\mathbb{P}^r, \mathcal{J}(n)) \cong H^i(\mathbb{P}^r, \mathcal{J}(n+1))$ for $i \geq 2$. By Serre Vanishing (See Thm III.5.2 in Hartshorne), $H^i(\mathbb{P}^r, \mathcal{J}(m))$ vanishes for large enough m, so we may assume $H^i(\mathbb{P}^r, \mathcal{J}(n)) = 0$ for $n \geq n_1$, $i \geq 2$.

It remains to deal with the first cohomology. We have an exact sequence:

$$H^0(\mathbb{P}^r, \mathcal{J}(n+1)) \xrightarrow{\alpha_n} H^0(H, \mathcal{F}(n+1)) \longrightarrow H^1(\mathbb{P}^r, \mathcal{J}(n)) \longrightarrow H^1(\mathbb{P}^r, \mathcal{J}(n+1)) \longrightarrow 0.$$

We know that $h^1(\mathbb{P}^r, \mathcal{J}(n+1)) \leq h^1(\mathbb{P}^r, \mathcal{J}(n))$, and indeed these are equal if the map from $H^1(\mathbb{P}^r, \mathcal{J}(n))$ to $H^1(\mathbb{P}^r, \mathcal{J}(n+1))$ is injective. This would imply that α_n is surjective, as its image is the kernel of the following, which we have shown to be the case. Hence we have two possibilities: α_n is surjective, or $h^1(\mathbb{P}^r, \mathcal{J}(n+1)) < h^1(\mathbb{P}^r, \mathcal{J}(n))$. Now we claim that if α_n is surjective, so is α_{n+1}. Indeed since α_n is surjective, so is the top arrow in the following:

$$\begin{tikzcd} H^0(\mathbb{P}^r, \mathcal{J}(n+1)) \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \arrow[r, two heads, "\alpha_n \otimes id"] \arrow[d] & H^0(H, \mathcal{F}(n+1)) \otimes H^0(H, \mathcal{O}_H(1)) \arrow[d, "\text{Surjective by induction}"] \\ H^0(\mathbb{P}^r, \mathcal{J}(n+2)) \arrow[r, "\alpha_{n+1}"] & H^0(H, \mathcal{F}(n+2)) \end{tikzcd}$$

It's now a trivial diagram chase to see α_{n+1} is surjective.

Thus as n increases, $h^1(\mathbb{P}^r, \mathcal{J}(n))$ decreases, until it stabilizes somewhere. Since $h^1(\mathbb{P}^r, \mathcal{J}(m)) = 0$ for large enough m (again by Serre vanishing), it stabilizes at zero. In particular we get $h^1(\mathbb{P}^r, \mathcal{J}(n)) = 0$ for $n \geq n_1 + h^1(\mathbb{P}^r, \mathcal{J}(n_1))$, as at each increase past n_1, the dimension drops by at least one, and hits zero, so the worst case estimate is the dimension at the start, $h^1(\mathbb{P}^r, \mathcal{J}(n_1))$.

On the other hand, we bound the dimension above, as we know from the Hilbert polynomial $h^1(\mathbb{P}^r, \mathcal{J}(n_1)) \leq h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n_1)) - q(n_1)$, which is independent of $\mathcal{J}$. We claim that our desired uniform bound is $n_0 = n_1 + h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n_1)) - q(n_1) + 1$. By the above remarks, certainly (1) holds for $n \geq n_0$, so we focus on (2). Consider the diagram:

$$\begin{tikzcd} H^0(\mathbb{P}^r, \mathcal{J}(n)) \arrow[r, "\eta"] & H^0(\mathbb{P}^r, \mathcal{J}(n+1)) \arrow[r, "\alpha_n"] & H^0(H, \mathcal{F}(n+1)) \\ & H^0(\mathbb{P}^r, \mathcal{J}(n)) \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \arrow[u, "\beta"] \arrow[r, "\gamma"', "\text{surjective for } n \geq n_0 - 1"] & H^0(H, \mathcal{F}(n)) \otimes H^0(H, \mathcal{O}_H(1)) \arrow[u, "\delta - \text{Surjective by induction.}"'] \end{tikzcd}$$

where the top row is exact. A diagram chase shows β is surjective, which completes the proof. ■

It is worth noting that this proof also holds, verbatim, when $\mathcal{J}$ is replaced with a coherent subsheaf of a fixed coherent sheaf on $\mathbb{P}^r$. This notion gives rise to the Quot scheme.

The above lemma now implies the more easily understood corollary:

Corollary: Let r be a non-negative integer, and $p(t)$ be a rational polynomial. Then there is an integer n_0 with the following property. Let $\mathbb{P}^r \times S \supset X \longrightarrow S$ be any flat family of subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(t)$, with $\psi: \mathbb{P}^r \times S \longrightarrow S$ the projection, and $\mathcal{J}_X$ the ideal sheaf of X in $\mathbb{P}^r \times S$. Then for any $n \geq n_0$, the following hold:

1) $\psi_* \mathcal{J}_X(n)$ is locally free of rank $q(n)$;
2) $R^i \psi_* \mathcal{J}_X(n) = 0$, $i \geq 1$;
3) the multiplication map $\psi_* \mathcal{J}_X(n) \otimes \psi_* \mathcal{O}_{\mathbb{P}^r \times S}(1) \longrightarrow \psi_* \mathcal{J}_X(n+1)$ is surjective;
4) for any morphism $\alpha: T \to S$, the natural map $\alpha^* \psi_* \mathcal{J}_X(n) \longrightarrow \varphi_* \mathcal{J}_Y(n)$ is an isomorphism, where $Y = X \times_S T \subset \mathbb{P}^r \times T$, and $\varphi: \mathbb{P}^r \times T \to T$ the projection.

The details involve some other results on flatness which we now skip. One can see now that the uniform n_0-theorem is essentially proven.

Constructing the Hilbert Scheme III.

This will be the meat of the first part. Let X be a closed subscheme of $\mathbb{P}^r$ with Hilbert polynomial $p(t)$. We set

$$q(n) = \binom{n+r}{r} - p(n)$$

to be the Hilbert polynomial of $\mathcal{I}_X$. Fix some integer n larger then the uniform bound we showed last time. Then we define the n^{th} Hilbert point of X to be the point $[H^0(\mathbb{P}^r, \mathcal{I}_X)]$ in the grassmannian $Gr(q(n), H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)))$.

One can check that this uniquely specifies X, as the lemma we proved last time says $\mathcal{I}_X$ is generated in degree n or more by $H^0(\mathbb{P}^r, \mathcal{I}_X(n))$. Moreover, the dimension of $H^0(\mathbb{P}^r, \mathcal{I}_X(m))$ is $q(m)$ for any $m \geq n_0$, so this means that the n^{th} Hilbert point of X lies in the subset $H \subset Gr(q(n), H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n))) = G$ whose points are vector spaces V such that the multiplication map

$$\mu_{m,V} : V \otimes H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m-n)) \longrightarrow H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m))$$

has the dimension of its image $q(m)$. By definition then V generates a homogeneous ideal with Hilbert polynomial $q(t)$, and therefore comes from a closed subscheme $X \subset \mathbb{P}^r$ with Hilbert polynomial $p(t)$.

So we see that $\mathrm{Hilb}_r^{P(t)}$ will have H as its closed points. So we need to construct a scheme structure on H and prove representability. Let $\xi : \mathbb{P}^r \times G \longrightarrow G$ be the projection, and $\mathcal{F} \hookrightarrow \xi_* \mathcal{O}_{\mathbb{P}^r \times G}(n)$ be the universal subsheaf on G. We then let

$$\mu_h : \mathcal{F} \otimes \xi_* \mathcal{O}_{\mathbb{P}^r \times G}(h-n) \longrightarrow \xi_* \mathcal{O}_{\mathbb{P}^r \times G}(h)$$

be the multiplication map. So now we denote by Σ_h the determinental locally closed subscheme of G defined by the condition $\operatorname{rank} \mu_h = q(h)$. That is, Σ_h is the intersection of the determinental subscheme which is defined by the vanishing of $(q(h)+1) \times (q(h)+1)$ minors, and the open subset where at least one $q(h) \times q(h)$ minor does not vanish. Since we have fixed n, and the above discussion wants this for all $h \geq n$, we have:

$$H = \bigcap_{h \geq n} (\Sigma_h)_{\text{red}}.$$

We would like to define $\mathrm{Hilb}_r^{P(t)}$ as the scheme-theoretic intersection of the Σ_h, however its not clear that this infinite intersection of locally closed subschemes give rise to a scheme-theoretic intersection at all.

Denote the finite intersections by $\Lambda_k = \bigcap_{n \leq h \leq k} \Sigma_h$. Its enough to show that these eventually stabilize as sets. Indeed if so, one can show that for some k_0 $\{\Lambda_k\}_{k \geq k_0}$ is a decreasing sequence of closed subschemes, which would stabilize by noetherian-ness.

We want to show that the set of closed points $|\Lambda_k|$ is constant for large k. To show this, we aim to show that for any closed point in G, a closed subscheme $Y_s \subset \mathbb{P}^r$ and an integer $N \geq n$, independent of s, s.t. $\operatorname{rank} \mu_{h,s} = h_{\mathcal{I}_{Y_s}}(h)$ for $h \geq N$.

If we have managed to do this, for $s \in \Lambda_{N+r}$, we have $\operatorname{rank} \mu_{h,s} = q(h)$ for $n \leq h \leq N+r$, hence we can show $q(h)$ and $h_{\mathcal{J}_{Y_s}}(h)$ coincide, and so $s \in \Lambda_h$ for all $h \geq N$, which proves the claim.

Now we can exhibit Y_s as the subscheme corresponding to the ideal generated by $V \subset H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n))$, where $[V] = s \in G$. To get the uniform integer, we will show Y_s is a fiber of a closed subscheme $\mathcal{Y} \subset \mathbb{P}^r \times G$. Set

$$\mathcal{F}_m = \mu_m(\mathcal{F} \otimes \xi_* \mathcal{O}_{\mathbb{P}^r \times G}(m-n)).$$

Then $\bigoplus_j \mathcal{F}_j$ is a graded sheaf of ideals in $\bigoplus_j \xi_* \mathcal{O}_{\mathbb{P}^r \times G}(j)$, so there is a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^r \times G}$ and an integer $N \geq n$ such that

$$\mathcal{F}_m = \xi_* \mathcal{I}(m) \quad \text{for} \quad m \geq N.$$

Denote by $\mathcal{Y}$ the subscheme of $\mathbb{P}^r \times G$ corresponding to $\mathcal{I}$. Some techical lemmas on flatness then proves the rest. This ends the construction of the scheme structure of $\mathrm{Hilb}_r^{p(t)}$.

The universal family $\mathcal{X} \longrightarrow \mathrm{Hilb}_r^{p(t)}$ is the restriction of $\mathcal{Y}$ to $\mathrm{Hilb}_r^{p(t)}$, and this turns out to be flat by standard results. As a consequence, any $\alpha : S \longrightarrow \mathrm{Hilb}_r^{p(t)}$ induces by pullback a flat family $X = \mathcal{X} \times_{\mathrm{Hilb}_r^{p(t)}} S \xrightarrow{f} S$ of subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(t)$, a proof of this uses the corollary from last time. Conversely any flat family of subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(t)$ will be, by the universal property of the Grassmannian, be the pullback of $\mathcal{X} \longrightarrow \mathrm{Hilb}_r^{p(t)}$ by a unique $\alpha : S \longrightarrow \mathrm{Hilb}_r^{p(t)}$.

Some remarks:

1) The above also holds for analytic families of proper subschemes. Hence every flat analytic family is locally the pullback of an algebraic one.
2) $\mathrm{Hilb}_r^{p(t)}$ is projective. Use the valuative criterion of properness.
3) $\mathrm{Hilb}_r^{p(t)}$ is connected. Proven in Hartshorne's thesis.