RESEARCH STATEMENT

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Contents

1. Introduction

My primary area of interest is in algebraic geometry. Roughly speaking, algebraic geometry is the study of solution sets to multiple polynomial equations in several variables (called algebraic varieties). Since polynomial rings are so well understood, it is natural to bring to bear algebraic techniques to tackle geometric questions about varieties, and vice versa. Additionally, algebraic geometry as a subject has many close connections with fields such as complex analysis, topology, number theory, and theoretical physics. My own research is focused on the interplay between homological algebra and the underlying geometry of the variety.

Classically, the detailed study of algebraic objects inevitably involves the study of modules (suitably defined) over them and their corresponding homological properties. For example, a very fruitful approach to studying groups is representation theory, which is the study of modules over the corresponding group ring. In algebraic geometry, a similar mantra has proven to be very effective, where the correct notion of module is known as a (quasi-)coherent sheaf. The homological properties of coherent sheaves in many situations is effectively captured in a specific category $\mathrm{D}^b(X)$, which we call the *bounded derived category* of X. To construct this category, one proceeds in several steps. First, consider the (abelian) category of coherent sheaves $\operatorname{coh} X$[1] and then pass to the category of bounded cochain complexes $\mathrm{Ch}^b(\operatorname{coh} X)$. One then forms $\mathrm{D}^b(X)$ by formally "inverting" quasi-isomorphisms (morphisms which induce an isomorphism on cohomology). The result is no longer an abelian category, but has a different structure known as a triangulation.

The derived category was originally discovered by Grothendieck and his student, Verdier, as a natural setting for homological algebra in algebraic geometry. However, since its discovery, the derived category has become an interesting object to study on its own. One of the most famous examples of this is Kontsevich's homological mirror symmetry program [Kon95], but besides this the study of $\mathrm{D}^b(X)$ has resulted in a very large number of (conjectural) connections to the intrinsic geometry of the variety X [Kuz16], as well as to other

[1]If the variety is affine, i.e. sits as a closed subset of $\mathbb{C}^n$, then $\operatorname{coh} X$ is the category of finitely generated modules over the ring of functions on X.

fields in mathematics, such as algebraic topology, symplectic topology, and representation theory. My interests in algebraic geometry lie in the study of this category, as well as its connections to other subfields of algebraic geometry, such as birational geometry and moduli theory, to name only a few.

The major area of my work so-far concentrates on understanding $\mathrm{D}^b(X)$ when X has mild singularities.[2] This is an aspect of the theory which has, up until somewhat recently, been mostly ignored. This is more then a question of internal interest, as (the derived categories of) singular varieties arise naturally in algebraic geometry, see for example the discussion in [Kaw17]. Hence it is important to understand the derived categories of singular varieties. However singularities can in general be very complicated, and the more complicated they are, the less overall control we have over $\mathrm{D}^b(X)$. This poses many theoretical challenges which are not present in the case when X is smooth.

2. Reconstruction of singular curves

My first project is to understand to what extent $\mathrm{D}^b(X)$ determines X. In other words, if we know $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ as triangulated categories, is it true that $X \cong Y$? When working with non-singular projective varieties, the answer is tied to the intrinsic geometry of X. Under these conditions, the famous "reconstruction theorem" of Bondal and Orlov in [BO01] states that the derived category is a complete invariant. It is worth noting that this result does not extend to all varieties, as there are examples of Calabi-Yau[3] varieties which are derived equivalent but not isomorphic. In [Spe20b] I showed that this reconstruction result is true for curves even when the curve has arbitrary singularities.

Before detailing this result, we need some context. My result shows that any equivalence between derived categories of curves induces an isomorphism of the two curves. The strategy is simple; from the functor $F : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$, produce an as-explicit-as-possible isomorphism $f : X \to Y$. This is far from possible in general, but there are a class of functors, called integral functors, where this is sometimes possible.

Given a functor $F : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$, it is a desirable property to be represented by an object on the product. Consider the diagram:

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y, \end{tikzcd}$$

with π_X and π_Y the projections. Given some object $\mathcal{K}$ in $\mathrm{D}^b(X \times Y)$, we ask if F is isomorphic to the triangulated functor defined by the pull-multiply-push formula:

$$\Phi_{\mathcal{K}}^{X \to Y}(-) = \pi_{Y*}(\pi_X^*(-) \otimes \mathcal{K}).$$

If so, we say that F is an integral functor with kernel $\mathcal{K}$. If it is also an equivalence, we say that it is a Fourier-Mukai functor.

For algebraic geometers, integral functors have taken center stage as the functors of primary interest, due both to utility and necessity, as seminal results of Lunts and Orlov show that any fully faithful functor $\mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ between the bounded derived categories

[2] X is said to be singular when the Jacobian matrix of its defining polynomials has less then its maximum possible rank.

[3] We say that a (smooth) variety is Calabi-Yau if the top exterior power of its cotangent bundle is trivial.

of projective varieties with a a right adjoint is an integral functor [LO10]. In particular, any equivalence is of this form.

Returning to the question of reconstruction; in the singular setting, the reconstruction theorem has been generalized to varieties with mild singularities (known as Gorenstein singularities) [Bal11, SdSSdS12]. The next most general type of singularity is that of Cohen-Macaulay type, and in fact in dimension one, singularities can be at worst Cohen-Macaulay. My contribution in [Spe20b] completes the reconstruction problem in dimension one by proving a reconstruction theorem for Cohen-Macaulay curves. More explicitly, a simple corollary to my work is the following.

Theorem 2.1. *Let X and Y be integral complex projective varieties such that $\dim X = 1$. Then $\mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$ if and only if $X \cong Y$.*

While this is interesting in its own right, the method of proof also yielded a few useful technical lemmas which are more widely applicable. For technical reasons, one often has to juggle with several different categories, $\operatorname{Perf} X \subset \mathrm{D}^b(X) \subset \mathrm{D}(\operatorname{Qcoh} X)$. By definition $\operatorname{Perf} X$ are those complexes in $\mathrm{D}^b(X)$ which are complexes of vector bundles, and $\mathrm{D}(\operatorname{Qcoh} X)$ is the same construction we detailed earlier but instead applied to quasi-coherent sheaves and dropping the requirement that the cochain complexes be bounded. Working with all three of these categories can be difficult, but in my work I was able to prove an extension/restriction type result for Fourier-Mukai functors:

Lemma 2.1. *Let X and Y be two projective varieties over the field k. Then given $\mathcal{K} \in \mathrm{D}(\operatorname{Qcoh}(X \times Y))$, the following are equivalent:*

(1) $\Phi_{\mathcal{K}}^{X \to Y} : \mathrm{D}(\operatorname{Qcoh} X) \xrightarrow{\sim} \mathrm{D}(\operatorname{Qcoh} Y)$,
(2) $\Phi_{\mathcal{K}}^{X \to Y} : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$,
(3) $\Phi_{\mathcal{K}}^{X \to Y} : \operatorname{Perf} X \xrightarrow{\sim} \operatorname{Perf} Y$,

Note that there are no assumptions on the singularities of X and Y, and the hypotheses that X and Y be projective can be relaxed somewhat. As indicated before, the strategy of proof is to construct an isomorphism from the functor $F : \mathrm{D}^b(X) \cong \mathrm{D}^b(Y)$, and from this construction, we can also determine the autoequivalence group of the derived category.

Corollary 2.1. *Let X be a curve of arithmetic genus different than one. Then*

$$\operatorname{Aut}(\mathrm{D}^b(X)) = \operatorname{Aut}(\operatorname{Perf} X) \cong \operatorname{Aut} X \ltimes (\operatorname{Pic} X \oplus \mathbb{Z}).$$

3. Indecomposability for some Cohen-Macaulay varieties

As in many areas of math it is often useful to decompose the objects you are studying into smaller, more manageable chunks. The derived category is no exception, and in this setting, the right notion of a decomposition is known as a semiorthogonal decomposition. My second project is an effort to understand semiorthogonal decompositions in the singular case. To discuss things in more detail, we define such a decomposition as follows.

Definition 3.1. *Let $\mathcal{T}$ be a triangulated category. A semiorthogonal decomposition of $\mathcal{T}$ is a collection of full admissible triangulated subcategories $\mathcal{A}_1, ..., \mathcal{A}_n$ of $\mathcal{T}$, called the components, such that*

(1) $\operatorname{Hom}_{\mathcal{T}}(A_j, A_i) = 0$ *for all $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$, $j > i$,*
(2) *The smallest triangulated subcategory containing all of the $\mathcal{A}_i$, $i = 1, ..., n$ is $\mathcal{T}$.*

In this scenario we write $\mathcal{T} = \langle \mathcal{A}_1, ..., \mathcal{A}_n \rangle$.

The existence of semiorthogonal decompositions often has drastic implications for the geometry of the variety in question, typically related to the rationality of the variety [Kuz16]. For an example from noncommutative geometry, consider when $\mathcal{A}_i \cong \mathrm{D}^b(\mathrm{Vect})$, the derived category of vector spaces, for all i. Under some additional assumptions, results of Bondal [Bon89] show that $\mathrm{D}^b(X)$ is then equivalent to the derived category of representations of a specific oriented quiver[4], which is explicitly constructed from the categories $\{\mathcal{A}_i\}$. On the more geometric side, it is conjectured for various X that if all of the components $\mathcal{A}_i$ are equivalent to $\mathrm{D}^b(Y_i)$, where Y_i is a variety with $\dim Y_i \leq \dim X - 2$, then X is rational [BB12]. In particular, one can see rather directly that the derived category of a cubic fourfold $W \subset \mathbb{P}^5$ admits a decomposition

$$\mathrm{D}^b(W) \cong \langle \mathcal{A}, \mathcal{O}_W, \mathcal{O}_W(1), \mathcal{O}_W(2) \rangle.$$

It is conjectured that $\mathcal{A}$ is equivalent to the derived category of a K3 surface precisely when W is rational.

While useful, semiorthogonal decompositions can be hard to produce without inspiration, but there are several cases which are well-understood. Of particular interest are the categories which are indecomposable, in the sense that they admit no nontrivial semiorthogonal decompositions. For example, the derived category of a Calabi-Yau variety or of a smooth projective curve (of positive genus). Results of Kawatani and Okawa in [KO15] provide the best criteria to date, linking the intrinsic geometry of the variety to indecomposability of its derived category. Specifically, for a smooth variety X one can define the canonical bundle ω_X, as the top exterior power of the cotangent bundle. It was shown in [KO15] that if the base locus[5] of the canonical bundle is small enough, then the derived category is indecomposable.

In forthcoming work [Spe20a], I was able to generalize this result of Kawatani and Okawa to the case of Cohen-Macaulay varieties. We work with the dualizing sheaf, similarly denoted ω_X, which serves as a replacement for the canonical bundle on singular varieties. Specifically, the following result is the main goal of the paper.

Theorem 3.1. *Suppose that X is a Cohen-Macaulay projective variety such that every connected component of $\mathrm{Bs}\,|\omega_X|$ is contained in an open subset on which ω_X is trivial. Then $\mathrm{Perf}\, X$ is indecomposable. If $\mathrm{Bs}\,|\omega_X|$ is empty, then $\mathrm{D}^b(X)$ is also indecomposable.*

I also hope to be able to show that all singular curves have an indecomposable category of perfect complexes, generalizing another result of Okawa which showed that smooth curves have indecomposable derived categories.

4. Future work

The above two projects that I have completed have supplied many further directions for investigation in the future. Although I have several questions in mind, the most immediately actionable in my opinion are the following.

Project 1: Reconstruction of singular varieties and derived equivalences. In light of Theorem 2.1, it is a natural question to ask if a similar result holds for varieties of higher dimension with possibly worse singularities. Considering only the former question for the moment, the proof contained in [Spe20b] will generalize to arbitrary dimensions so long as

[4]A quiver is a directed graph.

[5]The base locus of a vector bundle is the common vanishing set of its global sections.

the Cohen-Macaulay singularities are isolated, and there is a suitable notion of amplitude for the dualizing sheaf ω_X. There are a few competing notions of amplitude for a general sheaf, several among these seem to plausibly work for a generalization.

In addition, allowing worse singularities then just those of Cohen-Macaulay type seems to also be possible. The method of proof of Theorem 2.1 uses objects which are only defined for Cohen-Macaulay varieties. However by the work of [SdS09], it seems there are replacements which satisfy roughly the same properties. Thus it is plausible that an extension (granting the previous paragraph) to the case of arbitrary singularities is also possible.

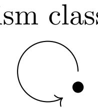
On the other hand, the question of when two nonisomorphic varieties can be derived equivalent is also a very interesting question. Such varieties are comparatively rare, but when the two varieties are smooth, many examples are known. The only known examples of derived equivalent and non-isomorphic singular varieties are contained in [MRV19], which provides examples of derived equivalent but nonisomorphic “singular abelian varieties”. In dimension two the only other class of varieties which are reasonable candidates are (singular) K3 surfaces, and in particular providing examples of derived equivalent singular K3 surfaces would be interesting for many reasons.

Project 2: The singularity category of curves. In my two projects detailed in the previous sections, we dealt with two main categories, $\mathrm{D}^b(X)$, and the subcategory of perfect objects, $\operatorname{Perf} X$. It is a well-known fact that X is regular if and only if these two categories agree, but in the singular case, it is interesting to understand the discrepancy between the two.

To this end, one can define the singularity category, given as the quotient $\mathcal{S}(X) = \mathrm{D}^b(X)/\operatorname{Perf} X$. Despite the fact that this category is closely connected to singularities (which a priori could be very complicated), it enjoys a number of useful properties [Orl06]. For example, if $U \subset X$ is an open subset which contains the singular locus, then $\mathcal{S}(U) \cong \mathcal{S}(X)$. If X is Gorenstein, then one can actually give a very explicit description of the singularity category. For worse singularities however, very little is known.

My proposed project is to explore the struture of the singularity category when X possesses non-Gorenstein singularities. Similar to previous projects, it makes sense to first investigate the structure for Cohen-Macaulay singularities. In particular, I propose to explore the structure of $\mathcal{S}(X)$ when X is the coordinate axes in affine 3-space, which is one of the simplest examples of a non-Gorenstein singularity of curves. From this I hope to extract some clues as to what the structure of the singularity category might be in more general situations.

Undergraduate research. While my work is very abstract, it supplies many interesting connections to more concrete subjects in which undergraduates can contribute in a meaningful way with only a course in algebra as a prerequisite. As mentioned earlier, when the derived category of a variety X admits a decomposition into pieces equivalent to the derived category of vector spaces, then the homological algebra encoded in $\mathrm{D}^b(X)$ can sometimes be effectively studied by instead working with representations of a directed quiver. A quiver is by definition a directed graph, and a representation of a quiver is simply a choice of vector space for each vertex, and a linear map between those spaces for each edge. The study of quiver representations is essentially a generalization of linear algebra. For example, the problem of understanding the (isomorphism classes of) representations of the quiver



is the same as classifying endomorphisms of a vector space under conjugation, i.e., Jordan canonical form.

As one can see, quivers are in many cases approachable and computational. Their study is full of interesting and explicit connections to geometry, but also to more surprising topics such as topological data analysis and combinatorics. In particular, there are many open questions that could be attempted by an undergraduate with proper supervision, requiring only a good understanding of linear algebra and some modest abstract algebra.

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