Research Statement

Dylan Spence

Introduction

My area of research is in algebraic geometry, specifically the study of algebraic varieties and schemes through the lens of noncommutative (or categorical) geometry via their *bounded derived category.* Roughly speaking, algebraic geometry is the study of solution sets to multiple polynomial equations in several variables (called algebraic varieties). Since polynomial rings are so well understood, it is natural to bring to bear algebraic techniques to tackle geometric questions about varieties, and vice versa. Additionally, algebraic geometry as a subject has many close connections with fields such as complex analysis, topology, number theory, and theoretical physics. My own research is focused on the interplay between homological algebra and the underlying geometry of the variety.

Classically, the detailed study of any algebraic object inevitably involves the study of modules (suitably defined) over them and their corresponding homological properties. For example, a very fruitful approach to studying groups is studying their representation theory; precisely the study of modules over the corresponding group ring. In algebraic geometry, a similar mantra has proven to be very effective, where the correct notion of module is known as a (quasi-)coherent sheaf. On a given variety X, the homological properties of coherent sheaves in many situations are effectively captured in a specific category $\mathrm{D}^b(X)$, which we call the *bounded derived category* of X. This category is formed by first taking the category of cochain complexes of coherent sheaves on X, and then formally "inverting" the quasi-isomorphisms – that is, morphisms which induce isomorphisms on cohomology.

The study of the derived category was pioneered by the works of Bondal and Orlov, and has since become a central object of study in algebraic geometry and related areas. Perhaps the most famous example of this is Kontsevich's homological mirror symmetry program [Kon95], which links both algebraic and symplectic geometry by a conjectural equivalence between the derived category of a Calabi-Yau variety and the Fukaya category of its "mirror". Another important example is Kuznetsov's homological projective duality, which provides convenient and geometrically interesting decompositions (called semiorthogonal decompositions) of $\mathrm{D}^b(X)$ for a large class of varieties X. Further, the study of $\mathrm{D}^b(X)$ from the viewpoint of these decompositions has resulted in a large number of (conjectural) connections to the intrinsic geometry of the variety X [Kuz16a], as well as to other fields in mathematics, such as algebraic topology, symplectic topology, and representation theory. My interests in algebraic geometry lie in the study of this category, as well as its connections to other subfields of algebraic geometry, such as birational geometry and moduli theory. In particular, I am interested in the study of derived categories of singular varieties, and interactions with moduli theory via Bridgeland stability conditions.

Bridgeland stability conditions originated in the work of Bridgeland in [Bri07, Bri08]. Roughly speaking, a stability conditions is a direct generalization of the notion of slope-stability of vector bundles on curves to objects of the derived category. Producing stability conditions is a difficult task in general, but many examples are now known. They allow one to construct moduli spaces of (semi)stable objects directly from the derived category. In particular, it is believed that the space of stability conditions contains a subset of so-called ample stability conditions, in which the variety X itself can be realized as a moduli space of stable objects. For this reason, stability conditions are viewed as the necessary data to "reconstruct" Calabi-Yau varieties from the derived category. Since their invention, however, Bridgeland stability conditions have provided many connections with other aspects of algebraic geometry, such as the minimal model program and Gromov-Witten theory, among others.

Research Objectives and Methods

Project 1: Derived categories of singular varieties.

Background. An aspect of derived categories which has, up until somewhat recently (see [Bal11, HRLMdS07, SdSSdS12, HRLMSdS09, KKS18]), been ignored is the case when X is singular. In addition to intrinsic interest, having a robust understanding of the derived categories of singular varieties is a desirable goal. In particular, according to Kawamata's famous *DK-hypothesis* ([Kaw17, Conjecture 1.2]), certain birational operations found in the minimal model program (MMP) should induce fully faithful functors between the respective derived categories. While the conjecture is stated in terms of smooth varieties only, the birational operations in the MMP can alter the original variety so that it is no longer smooth. Thus any full understanding of the interactions between the MMP and derived categories will need an understanding of the derived categories of singular varieties. This program also predicts that the derived categories of minimal varieties should satisfy some notion of indecomposability, which in many cases agrees with the nonexistence of semiorthogonal decompositions.

Such an understanding will need several new tools to overcome the technical difficulties that singularities impose on the derived category. For example, a better understanding of the so-called *singularity category*, defined as follows. To any variety X, one may attach two (related) categories of independent interest. The first being the derived category $\mathrm{D}^b(X)$ as we have already discussed, and the second being a full subcategory, $\operatorname{Perf} X$, which consists of complexes that are quasi-isomorphic to bounded complexes of locally free sheaves of finite rank. In the smooth case, these two categories are equivalent, but in the singular case, it is interesting to measure the discrepancy between these two. In [Orl06], Orlov introduced the notion of the *category of singularities* of a variety X as the Verdier quotient $\mathrm{D}^b(X)/\operatorname{Perf} X$. This category obeys several nice properties, in particular, when X possesses hypersurface (or complete intersection) singularities, the singularity category can be described explicitly via matrix factorizations in the sense of Eisenbud [Eis80, Orl06, BW15]. This category also features prominently in the homological mirror symmetry program for Fano varieties, where a version of it plays the same role as the derived category, but for Landau-Ginzburg models.

Another example is that of *categorical resolutions of singularities*. Geometrically, a resolution of singularities is a smooth variety $\tilde{X}$ together with a map $\tilde{X} \to X$ (satisfying reasonable properties), which is an isomorphism away from the singular locus of X. While resolutions always exist over the complex numbers, there are often many non-isomorphic resolutions, and an important question is when a variety X admits a resolution $\tilde{X}$ with certain desirable properties. In particular, so-called *crepant* resolutions form an important class of resolutions, but are rare and difficult to construct explicitly. From the viewpoint of the derived category however, given a singular variety X one may try to construct a "smooth" category $\mathscr{D}$ with a well-behaved functor $\mathscr{D} \to \mathrm{D}^b(X)$, which captures the homological properties of a geometric resolution of singularities. This idea has been explored in [Kuz08, KL15, vdB04] in varying levels of generality, but are universally referred to as categorical resolutions. The most interesting such construction is that of a *noncommutative crepant resolution* (NCCR) by Van den Bergh in [vdB04]. These resolutions, as indicated by the name, satisfy various homological properties which mimic that of a geometric crepant resolution. Moreover, these NCCRs often give rise to geometric resolutions via an explicit construction. Van den Bergh's NCCRs are now known to exist in a large number of cases [vVdB17, vVdB20], and are conjectured to be "derived-minimal", in the sense that any other categorical resolution should uniquely factor through the NCCR.

Overall, many basic questions about the derived categories of singular varieties, are still open; for example, classifying Fourier-Mukai partners of singular varieties in higher dimensions (which will need the theory of NCCRs), or producing examples of Bridgeland stability conditions on singular varieties. Answering these basic questions, and developing a robust set of tools to address the issues singularities pose, is the central thrust of this project.

Research. In [Spe20], I extended the reconstruction theorem of Bondal and Orlov to complex projective curves with arbitrary singularities; as a consequence of my work, the derived category is a complete invariant for complex projective curves. Additionally, in [Spe21], I was able to extend [KO15, Theorem 3.1] to varieties with Cohen-Macaulay singularities. As a corollary, algebraic curves of positive genus have indecomposable categories of perfect complexes, extending another theorem of Okawa [Oka11]. Combined, these results form a basis for the study of the derived categories of singular curves, but also require several new approaches to dealing with how singularities affect the derived category in arbitrary dimension, which I hope to apply in other situations.

Currently I am working to understand the structure of the singularity category in some examples. One fundamental question is to study the compact generators of this category. This has been solved in the hypersurface case by Dyckerhoff [Dyc11]. I am currently working to describe several examples of such compact generators in singularity categories where the singularities are not complete intersections.

Future research.

(1) Understand Bridgeland stability conditions for singular curves. Some work on this has already been completed [BK06], but the general case is still wide open. I expect similar techniques to those in *loc. cit.* will produce at least partial results.
(2) Begin an investigation of the derived categories of singular surfaces. In particular, Gorenstein K3 surfaces. Similar to the case of smooth K3 surfaces, there should be many interesting questions involving their (noncommutative) resolutions, Fourier-Mukai partners, and moduli of sheaves. Many examples of such K3 surfaces have been tabulated (see [IF00, Rei80] for example), and the first goal would be to understand their moduli of sheaves and more generally, any Bridgeland stability conditions which can be constructed.
(3) Develop a description of singularity categories for Cohen-Macaulay singularities in terms of matrix factorizations. Unlike the complete intersection case, no such description has been explicitly found, despite some indications that it should be possible ([EP15]). To begin I will frame my results concerning compact generation of the singularity category as above in terms of matrix factorizations. In fact I already have partial results in this direction.

Project 2: Bridgeland stability and homological projective geometry.

Background. In many areas of math it is often useful to decompose the objects you are studying into smaller, more manageable chunks. The derived category is no exception, and in this setting, the right notion of a decomposition is known as a semiorthogonal decomposition. While useful, semiorthogonal decompositions can be difficult to produce without inspiration, but Kuznetsov in [Kuz07] came up with a systematic way to produce geometrically interesting semiorthogonal decompositions of many varieties, known as homological projective duality (HPD).

The most general version of this theory, due to Perry in [Per19], takes as input a *Lefschetz category*, which is a (suitably enhanced) triangulated category with a fixed semiorthogonal decomposition which is compatible with a fixed autoequivalence. The output of this is a new Lefschetz category, built out of the same components as the original. The main result of homological projective duality states that the "interesting" semiorthogonal component in the (iterated) hyperplane sections of each category are equivalent. In several cases, this distinguished semiorthogonal component, often called the *Kuznetsov component*, controls much of the homological information of the variety in question and is typically very interesting to study on its own. For example, if W

is a smooth cubic fourfold, then its derived category admits a semiorthogonal decomposition arising from HPD. The corresponding Kuznetsov component $\mathrm{Ku}(W)$ is an example of a *K3 category*, that is, a category whose Hochschild homology agrees with that of a K3 surface, and whose Serre functor is given by $S_{\mathrm{Ku}(W)} = (-)[2]$, the shift-by-2 functor. In the above example, it is conjectured ([Kuz10]) that the cubic fourfold W is rational if and only if $\mathrm{Ku}(W)$ is equivalent to the derived category of an actual K3 surface.

The machinery of homological projective duality is expected to categorify large portions of classical projective geometry, and the expected resulting theory is commonly referred to as *homological projective geometry*. While this aspect of the theory is very new, its applications have already significantly impacted our understanding; for example in [KP21a], the authors produced the first example of two deformation equivalent, derived equivalent Calabi-Yau threefolds which are not birational [OR18, BCP20], settling the birational Torelli problem for Calabi-Yau threefolds in the negative.

Another recent invention in the theory of derived categories is that of Bridgeland stability conditions, as developed in [Bri07, Bri08]. A stability condition on a derived category is a direct generalization of slope-stability for vector bundles on curves, and allows one to study moduli of (semi)stable objects within the derived category. The space of stability conditions, $\mathrm{Stab}(X)$, under reasonable assumptions, naturally has the structure of a complex manifold, and comes with an obvious action by the group of autoequivalences of the derived category. This has been used in several places to help determine this group, which is of great interest for K3 surfaces and higher dimensional Calabi-Yau varieties.

Fixing some numerical data, the moduli space of (semi)stable objects with that data turn out to vary nicely within $\mathrm{Stab}(X)$. To be precise, there is a so-called *wall-and-chamber* structure on $\mathrm{Stab}(X)$, such that the (semi)stable objects with fixed numerical data stay constant in each chamber, but upon crossing a wall the moduli space changes in interesting ways. This phenomenon, known as *wall-crossing*, has been linked to the minimal model program [BM14], classical results such as Brill-Noether theory [Bay18], and conjecturally linked to both Gromov-Witten and Donaldson-Thomas theory, for example, [Tod18, PT09].

While Bridgeland stability conditions are clearly interesting objects to study, the first and most fundamental question is that of existence. Here things are difficult, and producing explicit examples is open in many interesting cases. Indeed, the only varieties where we have a reasonably complete understanding of the space of stability conditions are smooth projective curves [Mac07, Oka06]. For surfaces, we have many examples, and even a general construction [AB13], but only scattered and incomplete results about the space of such stability conditions [Li17, Bri08]. For threefolds and a handful of other higher-dimensional varieties, we have conjectures [BMT14], but very few results.

The combination of Kuznetsov and Perry's homological projective geometry with stability conditions also appears to have great potential. Here, much of the effort is devoted to producing and studying examples of K3 categories, for example, the Kuznetsov component of a cubic fourfold as above. K3 categories are interesting for many reasons, but one of the primary motivating factors is that certain moduli spaces of Bridgeland-stable objects in a K3 category are examples of higher-dimensional hyperkähler varieties, of which we have very few explicit constructions.

More generally, the study of Bridgeland stability conditions on arbitrary Kuznetsov components has proven to be a fruitful direction of inquiry. For example, various "categorical Torelli" theorems have been proven, as in [BMMS12], where the authors show that a moduli space of Bridgeland-stable objects in the Kuznetsov component of a cubic threefold is actually the Fano surface of lines on the threefold, from which it is possible to reconstruct its intermediate Jacobian. Hence the Kuznetsov component completely determines the isomorphism type of the cubic threefold. Another example is the work [BLM$^+$21], which develops the theory of stability conditions in families, and uses the theory to provide a new proof of the integral Hodge conjecture for cubic fourfolds. These techniques have been applied to Gushel-Mukai fourfolds as well [PPZ19, Per20] yielding a proof of the integral Hodge conjecture for Gushel-Mukai fourfolds, which was previously unknown.

Past and current research. I am currently involved in a project with A. Thomas Yerger, Yue Shi, and Aolong Li (graduate students at Indiana University Bloomington) to exhibit new stability conditions on some product quotient surfaces using [MMS09, Liu21b], and certain Calabi-Yau threefolds which can be exhibited as complete intersections, following [Liu21a, Li19].

Future research.

(1) I propose to study stability conditions on new examples of K3 categories. Several places to begin are *Debarre-Voisin* varieties [DV10] and (conjecturally) Küchle fourfolds [Kuz16b], as well as many weighted projective hypersurfaces [Kuz19]. Previous work to construct stability conditions on such categories involved finding an embedding into $\mathrm{D}^b(Y, \mathcal{A})$, where Y is some Fano threefold and $\mathcal{A}$ is a sheaf of algebras on Y. The techniques of homological projective geometry have produced such embeddings previously, and it is likely that similar arguments will work here.
(2) More generally, I will study stability conditions on Kuznetsov components. Beginning with investigating the presence of stability conditions on new examples. For example, it was shown in [KP21b] that the Kuznetsov component of a smooth complete intersection of a quadric and cubic in $\mathbb{P}^5$ has no Serre-invariant stability conditions, but it is not known if this category admits any stability conditions.
(3) Begin an investigation and develop a theory of how Bridgeland stability conditions interact with the constructions appearing in homological projective geometry. No direct work has been done on this, but several intermediate results are known [BLMS21, CP10].

Undergraduate Research

Another important aspect of academic inquiry is the training of future mathematicians. Besides teaching in a more traditional format, another important part of undergraduate education is undergraduate mentoring, broadly interpreted. A successful approach to undergraduate mentoring can vary widely from individual to individual, but typically takes the form of something similar to an independent study, or undergraduate research. I have been fortunate at Indiana University to be granted opportunities to be the primary mentor in both formats.

Firstly, regarding mentorship resembling an independent study, I am currently the program coordinator of the Indiana University Directed Reading Program (DRP). The DRP pairs an undergraduate with a graduate student for an independent study lasting the duration of the semester. This provides the undergraduate valuable experience learning in a less-structured environment, and also provides the graduate student valuable experience being a mentor. Before I was the coordinator, I successfully mentored two projects in algebraic geometry. I include more detail about this program and my involvement in my teaching statement.

Secondly, regarding mentorship in the form of undergraduate research, in the Spring of 2022, I will be running a project in the Indiana University Laboratory of Geometry (LogIU). This program has a similar goal as the DRP, but instead of an more-focused independent study, this program aims to have the students explore (with careful supervision) a specific question in higher mathematics with the goal of publishable research. I have proposed a project to study the lines on real cubic surfaces. Specifically, a smooth cubic surface over the complex number is classically known to contain exactly 27 lines, but over the real numbers, this no longer holds. Instead, due to a classical result of Schläfli, any real smooth cubic surface contains exactly 3,7,15, or 27 lines (!). Further, given a cubic surface over the complex numbers, there exist explicit formulae to determine the coefficients of the lines lying inside inside of it [MMZ21]. My proposed project is to, for real cubic surfaces, use a computer algebra software package to find explicit algebraic relations between the coefficients of the lines. This would provide the equations for a subvariety of the moduli space of real cubic surfaces, which would parametrize those surfaces containing a specified number of lines. While this may sound abstract, in practice this only requires the students to be comfortable with multivariable calculus and the use of a computer algebra software package.

My planned approach to supervising this project is as follows. First, the students must be exposed to the necessary background for the question to make sense. Rather then running them through an entire course in algebraic geometry however, I plan to instead gently introduce to them the theory of affine curves and surfaces first, through many examples and exercises, following the approach taken in the first 4 chapters of [GBB+13]. This would require only basic algebra (not abstract algebra) regarding polynomials and modest visualization skills (hence the requirement for calculus 3), aided by appropriate computer software. Strictly speaking, this knowledge would be all they need to complete the project, although more background on projective space and projective curves and surfaces will be given. Finally, an elementary proof of the 27 lines on a cubic surfaces will be presented, following [Rei88]. Secondly, the students will need to understand some portion of the work in [MMZ21], which, while lengthy, requires again mostly basic algebra. Of course, I will maintain very close supervision at this stage, as the goal of this portion of the project is not necessarily to understand [MMZ21] in its entirety, but rather the process that they used to obtain the various formulae and how their process can be better adapted to our situation. I estimate that the first two steps will take roughly two-thirds of the semester.

Finally, the students would need to transfer the equations from [MMZ21] and their modifications to a computer algebra software, and begin searching for algebraic relations between the coefficients. Since any cubic surface over the complex numbers contains 27 lines, any nontrivial relation at all would be interesting, but the key question that the students will need to address is when such a relation refers to the lines being *real*, that is, have all real coefficients.

Besides this project, there are many other projects for independent studies or undergraduate research that fit into the framework of my research interests above. Despite many aspects of derived categories being at the cutting edge of research, there are many important concrete examples that could be studied by motivated undergraduate students. For example, derived categories of quiver representations are an important object of study both in the structure of derived categories of varieties and for the theory of Bridgeland stability conditions. Specific aspects of their theory can be divorced from this context, and essentially reduced down to explicit problems in linear algebra, which, depending on the problem in question, may be tractable on a computer, if not by hand. Additionally, there are other nearby topics to my research, such as representation theory, algebraic topology, and combinatorial algebra. Within each of these fields there are numerous possible directions for a motivated undergraduate student to pursue an independent study or research project.

References

[AB13] Daniele Arcara and Aaron Bertram. Bridgeland-stable moduli spaces for K-trivial surfaces. *J. Eur. Math. Soc. (JEMS)*, 15(1):1–38, 2013. With an appendix by Max Lieblich.

[Bal11] Matthew Ballard. Derived categories of sheaves on singular schemes with an application to reconstruction. *Adv. Math.*, 227(2):895–919, 2011.

[Bay18] Arend Bayer. Wall-crossing implies Brill-Noether: applications of stability conditions on surfaces. In *Algebraic geometry: Salt Lake City 2015*, volume 97 of *Proc. Sympos. Pure Math.*, pages 3–27. Amer. Math. Soc., Providence, RI, 2018.

[BCP20] Lev A. Borisov, Andrei Căldăraru, and Alexander Perry. Intersections of two grassmannians in $\mathbb{P}^9$. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2020(760):133–162, 2020.

[BK06] Igor Burban and Bernd Kreussler. Derived categories of irreducible projective curves of arithmetic genus one. *Compos. Math.*, 142(5):1231–1262, 2006.

[BLM+21] Arend Bayer, Martí Lahoz, Emanuele Macrì, Howard Nuer, Alexander Perry, and Paolo Stellari. Stability conditions in families. *Publ. Math. Inst. Hautes Études Sci.*, 133:157–325, 2021.

[BLMS21] Arend Bayer, Marti Lahoz, Emanuele Macri, and Paolo Stellari. Stability conditions on kuznetsov components. *Annales Scientifiques de l'École Normale Supérieure*, May 2021.

[BM14] Arend Bayer and Emanuele Macrì. MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations. *Invent. Math.*, 198(3):505–590, 2014.

[BMMS12] Marcello Bernardara, Emanuele Macrì, Sukhendu Mehrotra, and Paolo Stellari. A categorical invariant for cubic threefolds. *Adv. Math.*, 229(2):770–803, 2012.

[BMT14] Arend Bayer, Emanuele Macrì, and Yukinobu Toda. Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities. *J. Algebraic Geom.*, 23(1):117–163, 2014.

[Bri07] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. of Math. (2)*, 166(2):317–345, 2007.

[Bri08] Tom Bridgeland. Stability conditions on $K3$ surfaces. *Duke Math. J.*, 141(2):241–291, 2008.

[BW15] Jesse Burke and Mark E. Walker. Matrix factorizations in higher codimension. *Trans. Amer. Math. Soc.*, 367(5):3323–3370, 2015.

[CP10] John Collins and Alexander Polishchuk. Gluing stability conditions. *Adv. Theor. Math. Phys.*, 14(2):563–607, 2010.

[DV10] Olivier Debarre and Claire Voisin. Hyper-kähler fourfolds and grassmann geometry. 2010(649):63–87, 2010.

[Dyc11] Tobias Dyckerhoff. Compact generators in categories of matrix factorizations. *Duke Mathematical Journal*, 159(2):223 – 274, 2011.

[Eis80] David Eisenbud. Homological algebra on a complete intersection, with an application to group representations. *Trans. Amer. Math. Soc.*, 260(1):35–64, 1980.

[EP15] Alexander I. Efimov and Leonid Positselski. Coherent analogues of matrix factorizations and relative singularity categories. *Algebra Number Theory*, 9(5):1159–1292, 2015.

[GBB$^+$13] Thomas Garrity, Richard Belshoff, Lynette Boos, Ryan Brown, Carl Lienert, David Murphy, Junalyn Navarra-Madsen, Pedro Poitevin, Shawn Robinson, Brian Snyder, and Caryn Werner. *Algebraic geometry*, volume 66 of *Student Mathematical Library*. American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 2013. A problem solving approach, IAS/Park City Mathematical Subseries.

[HRLMdS07] Daniel Hernández Ruipérez, Ana Cristina López Martín, and Fernando Sancho de Salas. Fourier-Mukai transforms for Gorenstein schemes. *Adv. Math.*, 211(2):594–620, 2007.

[HRLMSdS09] Daniel Hernández Ruipérez, Ana Cristina López Martín, and Fernando Sancho de Salas. Relative integral functors for singular fibrations and singular partners. *J. Eur. Math. Soc. (JEMS)*, 11(3):597–625, 2009.

[IF00] A. R. Iano-Fletcher. *Working with weighted complete intersections*, page 101–174. London Mathematical Society Lecture Note Series. Cambridge University Press, 2000.

[Kaw17] Yujiro Kawamata. Birational geometry and derived categories. arXiv:1710.07370, 2017.

[KKS18] Joseph Karmazyn, Alexander Kuznetsov, and Evgeny Shinder. Derived categories of singular surfaces. arXiv:1809.10628, 2018.

[KL15] Alexander Kuznetsov and Valery A. Lunts. Categorical resolutions of irrational singularities. *Int. Math. Res. Not. IMRN*, (13):4536–4625, 2015.

[KO15] Kotaro Kawatani and Shinnosuke Okawa. Nonexistence of semiorthogonal decompositions and sections of the canonical bundle. arXiv:1508.00682, 2015.

[Kon95] M. Kontsevich. Homological algebra of mirror symmetry. *Proceedings of the International Congress of Mathematicians*, 1995.

[KP21a] Alexander Kuznetsov and Alexander Perry. Categorical joins. *J. Amer. Math. Soc.*, 34(2):505–564, 2021.

[KP21b] Alexander Kuznetsov and Alexander Perry. Serre functors and dimensions of residual categories. arXiv:2109.02026, 2021.

[Kuz07] Alexander Kuznetsov. Homological projective duality. *Publ. Math. Inst. Hautes Études Sci.*, (105):157–220, 2007.

[Kuz08] Alexander Kuznetsov. Lefschetz decompositions and categorical resolutions of singularities. *Selecta Math. (N.S.)*, 13(4):661–696, 2008.

[Kuz10] Alexander Kuznetsov. Derived categories of cubic fourfolds. In *Cohomological and geometric approaches to rationality problems*, volume 282 of *Progr. Math.*, pages 219–243. Birkhäuser Boston, Boston, MA, 2010.

[Kuz16a] Alexander Kuznetsov. Derived categories view on rationality problems. *Rationality Problems in Algebraic Geometry*, page 67–104, 2016.

[Kuz16b] Alexander Kuznetsov. Küchle fivefolds of type c5. *Mathematische Zeitschrift*, 284(3-4):1245–1278, Jun 2016.

[Kuz19] Alexander Kuznetsov. Calabi–Yau and fractional Calabi–Yau categories. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2019(753):239–267, 2019.

[Li17] Chunyi Li. The space of stability conditions on the projective plane. *Selecta Math. (N.S.)*, 23(4):2927–2945, 2017.

[Li19] Chunyi Li. On stability conditions for the quintic threefold. *Invent. Math.*, 218(1):301–340, 2019.

[Liu21a] Shengxuan Liu. Stability condition on Calabi-Yau threefold of complete intersection of quadratic and quartic hypersurfaces. arXiv:2108.08934, 2021.

[Liu21b] Yucheng Liu. Stability conditions on product varieties. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2021(770):135–157, 2021.

[Mac07] Emanuele Macrì. Stability conditions on curves. *Math. Res. Lett.*, 14(4):657–672, 2007.

[MMS09] Emanuele Macrì, Sukhendu Mehrotra, and Paolo Stellari. Inducing stability conditions. *J. Algebraic Geom.*, 18(4):605–649, 2009.

[MMZ21] Stephen McKean, Daniel Minahan, and Tianyi Zhang. All lines on a smooth cubic surface in terms of three skew lines. *New York J. Math.*, 27:1305–1327, 2021.

[Oka06] So Okada. Stability manifold of $\mathbb{P}^1$. *J. Algebraic Geom.*, 15(3):487–505, 2006.

[Oka11] Shinnosuke Okawa. Semi-orthogonal decomposability of the derived category of a curve. *Adv. Math.*, 228(5):2869–2873, 2011.

[OR18] John Christian Ottem and Jørgen Vold Rennemo. A counterexample to the birational Torelli problem for Calabi–Yau threefolds. *Journal of the London Mathematical Society*, 97(3):427–440, 2018.

[Orl06] D. O. Orlov. Triangulated categories of singularities, and equivalences between Landau-Ginzburg models. *Mat. Sb.*, 197(12):117–132, 2006.

[Per19] Alexander Perry. Noncommutative homological projective duality. *Adv. Math.*, 350:877–972, 2019.

[Per20] Alexander Perry. The integral Hodge conjecture for two-dimensional Calabi-Yau categories. arXiv:2004.03163, 2020.

[PPZ19] Alexander Perry, Laura Pertusi, and Xiaolei Zhao. Stability conditions and moduli spaces for Kuznetsov components of Gushel-Mukai varieties. arXiv:1912.06935, 2019.

[PT09] R. Pandharipande and R. P. Thomas. Curve counting via stable pairs in the derived category. *Invent. Math.*, 178(2):407–447, 2009.

[Rei80] Miles Reid. Canonical 3-folds. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 273–310. Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.

[Rei88] Miles Reid. *Undergraduate algebraic geometry*, volume 12 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1988.

[SdSSdS12] Carlos Sancho de Salas and Fernando Sancho de Salas. Reconstructing schemes from the derived category. *Proc. Edinb. Math. Soc. (2)*, 55(3):781–796, 2012.

[Spe20] Dylan Spence. Reconstruction of projective curves from the derived category. arXiv:2002.07961, 2020.

[Spe21] Dylan Spence. A note on semiorthogonal indecomposability of some Cohen-Macaulay varieties. arXiv:2104.13331, 2021.

[Tod18] Yukinobu Toda. Curve counting theories on Calabi-Yau 3-folds: approach via stability conditions on derived categories [translation of MR3244103]. *Sugaku Expositions*, 31(2):199–229, 2018.

[vdB04] Michel van den Bergh. Non-commutative crepant resolutions. In *The legacy of Niels Henrik Abel*, pages 749–770. Springer, Berlin, 2004.

[vVdB17] Špela Špenko and Michel Van den Bergh. Non-commutative resolutions of quotient singularities for reductive groups. *Invent. Math.*, 210(1):3–67, 2017.

[vVdB20] Špela Špenko and Michel Van den Bergh. Non-commutative crepant resolutions for some toric singularities I. *Int. Math. Res. Not. IMRN*, (21):8120–8138, 2020.