Here we will aim to provide a relatively introductory course on moduli spaces, concentrating on a handful of major examples. Roughly speaking, the outline is as follows:

1) The Hilbert Scheme: Construction, examples.
2) Stable Curves: Basics - mostly for the Kontsevich space $\overline{M}_{g,n}$.
3) Deformation Theory
4) Moduli of Curves: Algebraic spaces, Deligne-Mumford Stacks, The Kontsevich space.
5) Applications (if time).

I will give most of, if not all, the talks, but of course people are welcome to volunteer. There will also be interesting side topics as we go.

Today I will begin with the basics: What is a moduli space? Why do we care? And I will present some ideas from algebraic geometry that we didn't get to over the summer, namely the Hilbert polynomial and flatness.

## The What/Why of Moduli

It is a deep and mysterious fact in algebraic geometry that "most" algebraic objects (namely varieties/schemes, sheaves, even morphisms) can be given a "reasonable" (say, finite-type) scheme structures. In addition, such objects appear naturally in answers to concrete questions, as well as being interesting in their own right, and providing insight to the classification question.

The notion of a parameter space should be clear: given a set of objects, we would like to find a scheme such that its closed points are in one-to-one correspondence with the elements of the set. If the collection of objects is considered up to isomorphism, then we call it a moduli space.

The distinction seems rather uninteresting, yet can have a profound impact. Take the following example:

Smooth Quadrics in $\mathbb{P}^3$

Let $X$ be a quadric in $\mathbb{P}^3$. That is, a hypersurface defined via a single homogeneous quadratic polynomial in $x_0, \dots, x_3$. It is clear that this polynomial can be represented by a matrix of coefficients $Q$, so that:

$$f(x_0, \dots, x_3) = (x_0, x_1, x_2, x_3) \cdot Q \cdot (x_0, x_1, x_2, x_3)^T.$$

and such a $Q$ is naturally taken to be symmetric. What's not obvious is that $X$ is smooth $\Leftrightarrow$ $Q$ has full rank, and so the smooth quadrics cut out a positive dimensional subvariety in $PGL(4,k)$ (note they do not form a group). This is the parameter space.

However it's a fact of linear algebra that we can change coordinates to get $f = x_0 x_2 - x_1 x_3$, and so up to isomorphism, there is only one quadric surface.

Another Example is certainly one that every student of AG has heard of.

## The Dual Curve.

Let $X$ be a smooth curve in $\mathbb{P}^2$ and $p \in X$ a point. Then the tangent line at $p$ $L_p = T_pX$ is given by a degree one curve: $L_p = V(a(x-p_x) + b(y-p_y) + c(z-p_z))$ and the coefficients are the partials of the defining polynomial $f$ of $X$ (evaluated at $p$). Thus we have a map: $X \to (\mathbb{P}^2)^* \cong \mathbb{P}^2$ by:

$$p \mapsto [\partial f/\partial x(p) : \partial f/\partial y(p) : \partial f/\partial z(p)].$$

The image sweeps out another (possibly singular) curve in $\mathbb{P}^2$, called the dual curve.

The above example of course generalizes to hypersurfaces of any fixed degree in any $\mathbb{P}^n$, by sending the hypersurface $X_\alpha$ to the point $\alpha \in \mathbb{P}^N$ consisting of the coefficients of its defining equation $f_\alpha$. If $X_\alpha$ has degree $d$, then $N = \binom{n+d}{d} - 1$.

The parameter space above comes with a so-called "tautological family". That is, define a subset $\mathcal{F} \subset \mathbb{P}^N \times \mathbb{P}^n$ defined by $\sum a_I x^I = 0$, $I$ a multi-index of norm $d$. Then $\mathcal{F}$ is a hypersurface in the product, and comes with a natural projection $\mathcal{F} \to \mathbb{P}^N$, such that the fiber over $\alpha \in \mathbb{P}^N$ is exactly the curve with coefficients given by $\alpha$. We will see in maybe the next lecture that this is the first example of a Hilbert scheme.

The Hilbert scheme will be our first main topic, but we should say something about our eventual goal. We wish to obtain a relatively complete description of the moduli space $M_{g,n}$ of stable $n$-pointed curves of genus $g$.

## Moduli of Elliptic Curves

By definition, $M_g$ is the set of isomorphism classes of smooth curves of genus $g$. Given a genus one smooth curve (i.e. an elliptic curve), it is well known that it has the plane model

$$y^2 = 4x^3 - g_2 x - g_3.$$

It is also known that the quantity

$$j = 1728\, g_2^3 / (g_2^3 - 27 g_3^2)$$

depends only on the isomorphism class of the curve and so induces a one-to-one map

$$j : M_1 \to \mathbb{C}.$$

For technical reasons, there are many issues with families over $M_1$.

Let us now end this section by explaining what we mean by a moduli problem. A moduli problem is one of classification. While specifying a parameter space is a description of how the objects can vary, a moduli space captures isomorphism classes of objects and all the ways they can vary.

Given a space $B$, we can form the set $S(B)$ of isomorphism classes of families over $B$. Further, given a family $E \xrightarrow{\pi} B$, and a map $f: B' \to B$, then we can naturally get a family over $B'$ by taking the fiber product $E \times_B B'$, i.e., to some point $x \in B'$ assign the object parametrized by $f(x) \in B$.

We require that this construction actually gives a family, and the iso. class of $E \times B'$ depends only on the iso. class of $E$. This is essentially saying that varying the objects is invariant under base change. Such a new family is called the pullback family. Thus we get a map $S(f): S(B) \to S(B')$, called the moduli functor.

If this moduli functor is representable, the object representing it is called the moduli space.

## The Hilbert Polynomial

The Hilbert polynomial is a very interesting object in it's own right, and turns out to measure many interesting invariants of projective varieties.

Thm: Let $M$ be a finitely generated graded module over the ring $S = \mathbb{C}[x_0, \dots, x_n]$, that is, we are given a decomposition $M = \bigoplus_{k=k_0}^{\infty} M_k$ with $f \cdot M_k \subseteq M_{k+d}$ for any $f \in S$ of degree $d$. Then there is a polynomial $P_M(t)$ with $\deg P_M(t) \leq n$ with rational coefficients such that:

$$\dim_{\mathbb{C}} M_k = P_M(k)$$

for all $k$ sufficiently large.

In particular for a projective variety $X = \operatorname{Proj} S/I$, we can take $M = S/I$ and define the Hilbert polynomial of $X$ as $P_X(t) = P_M(t)$.

[Note in general we define the Hilbert polynomial of a coherent sheaf $\mathcal{F}$ as $\chi(\mathcal{F}(n))$, and we will have a corresponding Hilbert polynomial $P_{\mathcal{F}}(t)$. For a variety we take $\mathcal{F} = \mathcal{O}_X$. The result known as Serre vanishing tells us $H^i(X, \mathcal{O}_X(k)) = 0$ $\forall i \geq 1$ for $k$ large enough, so this is the "right" generalization.]

Examples:

1) Take $X = \mathbb{P}^n_{\mathbb{C}}$. Then

$$\dim_{\mathbb{C}} M_k = \#\{\text{monomials of degree } k \text{ in } n+1 \text{ variables}\} = \binom{n+k}{k}$$

$$= \frac{(k+n)\cdots(k+1)}{n!} = \frac{k^n}{n!} + \text{stuff}.$$

So here the Hilbert function is the Hilbert polynomial.

2) Let $X = V(f)$ be a degree $d$ Hypersurface in $\mathbb{P}^n$. Then we have an exact sequence:

$$0 \longrightarrow S_{k-d} \xrightarrow{\cdot f} S_k \xrightarrow{\pi} M_k \longrightarrow 0,$$

where $M = S/I$. Then $\dim M_k = \dim S_k - \dim S_{k-d} = \binom{k+n}{k} - \binom{k-d+n}{k-d}$ by (1).

Thus $P_X(t) = d\,\dfrac{t^{n-1}}{(n-1)!} + \text{stuff}$. Moreover one can characterize hypersurfaces among varieties by having this form of the Hilbert polynomial.

3) Another interesting example is where $X = \{a_1, \dots, a_s\} \subset \mathbb{P}^n$ is a finite set of points. We may assume that all points lie in $U_0$. Then for all $k$, we have the exact sequence

$$0 \longrightarrow I(X)_k \longrightarrow S_k \xrightarrow{\phi} \bigoplus_{i=1}^{s} \mathbb{C}$$

where $\phi(f) = \left(\frac{f}{x_0^k}(a_1), \dots, \frac{f}{x_0^k}(a_s)\right)$. In general for $k < s$, the Hilbert function will vary depending on the position of the points, but for $k \geq s$, we can take a collection of $k$ polynomials, not vanishing at all $a_i$, and construct a new polynomial with $f_i(a_j) = \lambda_i \delta_{ij}$, $\lambda_i \in \mathbb{C}$. Then $\phi$ will be surjective, and hence:

$$h_X(k) = \dim S_k - I(X)_k = s \quad \forall\ k \geq s.$$

This hints at a nice characterization of the Hilbert polynomial, which we now state:

Thm: Let $X \subset \mathbb{P}^n$ be a projective variety of dimension $r$ and degree $d$. Then its Hilbert polynomial has the form $P_X(t) = (d/r!)\, t^r + (\text{lower order terms})$.

[Recall the degree of a projective variety of dimension $r$ is the number of points in the intersection $X \cap L$, where $L$ is a generic linear subspace of dimension $n-r$ in $\mathbb{P}^n$.]

While the leading term is clearly of importance, so is the constant term, which is used to define the so-called arithmetic genus: $p_a = (-1)^r (P_X(0) - 1)$. Hopefully one now sees that varieties with the same Hilbert polynomial share many properties of interest.

One reason that the Hilbert scheme is the first family we will look at is that it is an alternative characterization of family. In particular we want the notion of flatness (which is a technical condition), but over a reduced base scheme the condition of flatness and constancy of the Hilbert polynomial are equivalent!