Kähler Differentials

Let $f: A \to B$ be a ring homomorphism.

Def: An A-derivation of B into a B-module M is a map $d: B \to M$ s.t.

1) $d(b_1 + b_2) = db_1 + db_2$
2) $d(bb') = bdb' + b'db$
3) $da = 0 \quad \forall a \in A$.

There is a universal differential $d: B \to \Omega^1_{B/A}$, into the module of Kähler differentials. For various constructions and details, see Hartshorne or EGA.

Some properties:

- If B is a f.g. A-algebra, then $\Omega^1_{B/A}$ is f.g. as a B-module.
- If we base change:

$$\left.\begin{array}{ccc} B & \longrightarrow & B\otimes_A A' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}\right\} \Rightarrow \Omega^1_{B'/A'} = \Omega^1_{B/A} \otimes_B B'$$

- If $S \subset B$ is a multiplicative subset, then $A \to B \to S^{-1}B$ gives $\Omega^1_{S^{-1}B/A} = S^{-1}\Omega^1_{B/A} = S^{-1}B \otimes_B \Omega^1_{B/A}$.
- If (B, m) is local and $A \cong B/m$ a field, then the fiber
$$(\Omega^1_{B/k}) \otimes_B (B/m) \cong m/m^2,$$
the Zariski cotangent space at m.
- If $I \subset B$ is an ideal and $C = B/I$, then we get an exact sequence of C-modules:
$$I/I^2 \longrightarrow \Omega^1_{B/A} \otimes_B C \longrightarrow \Omega^1_{C/A} \longrightarrow 0.$$
- (Thm 8.6a, Hartshorne). If L/k is a field extension, then $\dim \Omega^1_{L/k} \geq \mathrm{tr.deg}\, L/k$, with equality iff L/k is separably generated. So if $[L:k] < \infty$, $\Omega^1_{L/k} = 0 \iff L/k$ is separable.
- (Prop. 8.3a, Hartshorne). If $A \to B \to C$, we get an exact sequence of C-modules:
$$\Omega^1_{B/A} \otimes_B C \longrightarrow \Omega^1_{C/A} \longrightarrow \Omega^1_{C/B} \longrightarrow 0.$$
- (Prop. 8.11, Hartshorne). If $X \xrightarrow{f} Y \xrightarrow{g} Z$ a morphism of schemes, we get an exact sequence of $\mathcal{O}_X$-modules: $f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$.

Note that using the interplay with localizations, $\Omega_{X/Y}$ makes sense for any morphism of schemes $f: X \to Y$. If f is locally of finite type, $\Omega_{X/Y}$ is a coherent sheaf on X.

Def: A morphism LoFT $f: Y \to X$ is unramified if $\Omega^1_{Y/X} = 0$.

One such example is surjective maps of rings $A \twoheadrightarrow B$ (think: d vanishes on A), and hence all closed immersions are unramified. Another is $\mathrm{Spec}\, L \to \mathrm{Spec}\, k$, where L/k is finite; this will be unramified iff L/k is separable.

Prop: Let $f: Y \to X$ be LoFT. TFAE:

1) f is unramified;
2) $\forall y \in Y$, $x = f(y)$, we have $m_x \cdot \mathcal{O}_{Y,y} = m_y$ (have $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$), & $k(x)/k(y)$ is finite and separable;
3) $\forall x \in X$, the fiber Y_x is a discrete space $Y_x = \coprod_{y \in Y_x} \mathrm{Spec}\, k(y)$ & $k(x)/k(y)$ is finite and separable.
4) For any geometric point $\bar{x}$ ($= \mathrm{Spec}\,\overline{k(x)} \to X$), the geometric fiber $Y_{\bar{x}}$ is a disjoint union $\coprod \mathrm{Spec}\, Y_{\bar{x}}$.
5) The diagonal $Y \to Y \times_X Y$ is an open embedding.

(Must-do: no credit though).

Exercise: A finite field extension L/k is separable iff $L \otimes_k L$ has no nilpotents.

Example: $A = \mathbb{F}_p(t)$, $B = \mathbb{F}_p(\sqrt[p]{t}) = \mathbb{F}_p(s)$. Then $\operatorname{Spec} B = Y \to X = \operatorname{Spec} A$ is ramified as even though $Y \hookrightarrow Y \times_X Y$ is a closed immersion ($B \otimes_A B \twoheadrightarrow B$), it is not an open immersion. The kernel is $(s \otimes 1 - 1 \otimes s)$, and this element is nilpotent (char p!).

Lemma: Let (B, m) be local, and $A = k$ a field. Then $\Omega^1_{B/A} = 0$ iff $m = 0$ (B is a field), and B/k is finite and separable.

Proof: ($\Leftarrow$). Thm 8.6a of Hartshorne.
($\Rightarrow$). Assume $\Omega^1_{B/k} = 0$. We first show Krull-dim $B = 0$. Indeed suppose we had a nonmaximal prime $p \subset B$. Let $S = B \setminus p$. Then $S^{-1}\Omega^1_{B/k} = \Omega^1_{B_p/k} = 0$. Hence we have $k \hookrightarrow B_p \to B_p/pB_p \Rightarrow p/p^2 \to \Omega^1_{B_p/k} \otimes_{B_p} k(p) \twoheadrightarrow \Omega^1_{k(p)/k} \to 0$.

$$\underbrace{\Omega^1_{B_p/k} \otimes_{B_p} k(p)}_{=0} \Rightarrow \underbrace{\Omega^1_{k(p)/k}}_{=0}$$

Hence $k(p)/k$ is a finite separable extension. Yet tr. deg. $k(p)/k > 0$, and this is a contradiction. Hence B is a local artinian ring. If we show $m = 0$, we are done.
Since $\Omega^1_{B/k} = 0$, $K = \ker(B \otimes_k B \to B)$ is nilp. Running the same argument above, $(B/m)/k$ is finite & separable. As a k-vector space:

$$B \otimes_k B = \underbrace{(m \otimes m) \oplus (m \otimes k(m)) \oplus (k(m) \otimes m)}_{K \text{ - consists of nilpotent elmts.}} \oplus (k(m) \otimes k(m))$$

Hence $K/K^2 \neq 0$, and this is a contradiction. ∎

Proof of Prop:
(a) $\Leftrightarrow$ (b): Let $x \in X$. Base change $\Rightarrow$ $\Omega^1_{Y/X} \otimes_{\mathcal{O}_X} k(x) = \Omega^1_{Y_x/x}$. Let $y \in Y_x$. Then apply the lemma with $k = k(x)$, $B = \mathcal{O}_{Y_x,y}$. Then $\Omega^1_{Y_x/x} = 0 \; \forall x \Leftrightarrow \Omega^1_{x/y} = 0 \Leftrightarrow \mathcal{O}_{Y_x,y}$ is a finite separable field extension of $k(x)$.

(b) $\Leftrightarrow$ (c): Clear.

(c) $\Rightarrow$ (d): Note if L/k is finite separable, $L = k[t]/(f)$, and so $L \otimes_k \bar{k} = \bar{k}[t]/(f)$ and f splits into $\deg L/k$ linear distinct factors. Hence Chinese Remainder gives $L \otimes_k \bar{k} \cong \bar{k} \oplus \cdots \oplus \bar{k}$ $\deg L/k$ summands. Applying this to our context gives the claim.

(d) $\Rightarrow$ (a): Note $\overline{k(x)} \otimes_{k(x)} \Omega^1_{Y/X} = \Omega^1_{Y_{\bar{x}}/\overline{k(x)}}$, and $Y_{\bar{x}} = \coprod \overline{k(x)}$.

(a) $\Leftrightarrow$ (e): We may replace $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$ and so we have:

$$I \to B \otimes_A B \twoheadrightarrow B,$$

with $\Omega^1_{B/A} = I/I^2$. WTS $I/I^2 = 0 \Leftrightarrow \Delta : Y \to Y \times_X Y$ an open embedding. Since Δ is already a closed immersion, note:

$$\operatorname{Im}(\Delta) = \{ z \in \operatorname{Spec} B \otimes_A B \mid \underbrace{I_z \subsetneq \mathcal{O}_z}_{\mathcal{O}_z/I_z \neq 0 \;\Leftrightarrow\; z \in \operatorname{Supp}(B \otimes_A B/I)} \}$$

Now we use the fact that if $\varphi^\#$: Spec $D \to$ Spec C is a closed embedding, with ideal sheaf $\mathcal{J}$, then $\varphi^\#$ is an open immersion $\iff$ $\mathcal{J}_z = 0$ or C_z. Applying this to our case, a short argument shows the result. ▨

Corollary: If $f: Y \to X$ is unramified and separated, then $\Delta: Y \to Y \times_X Y$ is open and closed.

Example

1) $\underset{A}{\underset{\parallel}{k}} \longrightarrow \underset{B.}{\underset{\parallel}{k[t]/t^2}}$ has $\Omega^1_{B/A} \neq 0$. Indeed $\partial/\partial t \in \mathrm{Der}(k[t]/t^2)$.

2) Spec $(\mathbb{Z}[t]/(t^2+1)) \longrightarrow$ Spec $\mathbb{Z}$. Pick $(p) \in$ Spec $\mathbb{Z}$. Then $Y_{(p)} =$ Spec $\mathbb{F}_p[t]/(t^2+1)$

$$= \begin{cases} \mathbb{F}_p \times \mathbb{F}_p & \text{unramified if } p \equiv 1 \bmod 4 \\ \mathbb{F}_{p^2} & \text{unramified if } p \equiv 3 \bmod 4 \\ \mathbb{F}_2[y]/y^2 & \text{ramified. if } p = 2. \end{cases}$$

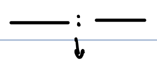
3) Double cover: Spec $\mathbb{C}[x,x^{-1}] \longrightarrow$ Spec $\mathbb{C}[x,x^{-1}]$ by $x^2 \longleftarrow x$. Unramified!

Def: $f: Y \to X$ is étale if f is flat and unramified (in particular, LoFT).

Note this is really a pointwise condition on Y.

Examples:

1) Spec $\mathbb{Z}[1/2][x]/(x^2+1) \to$ Spec $\mathbb{Z}[1/2]$ as we are throwing away $p=2$, and so is flat and unramified.

2) The map  $\Big\}$ is étale! an example of a nonseparated étale map.

Def: A morphism Spec $C \to$ Spec A is standard étale if $B = A[t]/\underset{\text{monic.}}{p(t)}$, and $C = B_b$, where $p'(t)$ is invertible.

Exercise 7 (2): Let $f: Y \to X$ be étale. Prove that locally on Y + X, f is standard. That is, $\forall y \in Y$, $x = f(y)$, $\exists$ affine opens $y \in V$, $x \in \mathcal{U}$, s.t. $f|_V : V \to \mathcal{U}$ is standard étale.

Exercise 8 (2): Hartshorne Ex. III. 10.4. (implies that f is étale at x iff $f_*: T_x X \xrightarrow{\sim} T_y Y$)

The above exercise shows that over $\mathbb{C}$, taking completions $\iff$ allowing holomorphic functions. As a remark, the condition for $f: Y \to X$ the conditions of being unramified and of being flat are open conditions.

Example: Étale Coordinate System:

Take $X \to$ Spec k, $k = \bar{k}$ a regular variety. Take $x \in \mathcal{U} \subset X$, and $f_1, \dots, f_n \in \mathcal{O}_{X,x}$, forming a basis for m_x/m_x^2. Now we can find an open subset of $\mathcal{U}$ w/ $(f_1, \dots, f_n): \mathcal{U} \to \mathbb{A}^n_k$ regular, taking $x \mapsto 0$. When we take completions: $\hat{\mathcal{O}}_{\mathbb{A}^n_k, 0} \cong k[[t_1, \dots, t_n]] \xrightarrow{\sim} k[[f_1, \dots, f_n]] \cong \hat{\mathcal{O}}_{X,x}$.
Hence the map $\mathcal{U} \to \mathbb{A}^n_k$ is locally étale around x, and is the best approximation to a neighborhood.

Exercise 9: Prove Prop. 3.19 in EC.

Exercise 10: Hartshorne Ex. III.10.6

Prop:
a) An open immersion is étale,
b) Composition of étale maps is étale,
c) Any base change of an étale map is étale.

Proof: Prop 3.3 in EC. ∎

Corollary: Consider morphisms $Y \xrightarrow{g} X \xrightarrow{f} S$. If $f\circ g$ is étale and f unramified, then g is étale.

Proof: Consider:

$$\begin{tikzcd}
Y \arrow[dr, dashed, "\Gamma_g"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "id"'] & & \\
& Y\times_S X \arrow[r, "p_2"] \arrow[d] & X \arrow[d, "f"] \\
& Y \arrow[r, "fg"] & S
\end{tikzcd}$$

Suffices to show p_2 is étale (pullback of fg ⇒ done), and the graph Γ_g is étale.

Now take:

$$\begin{tikzcd}
& & X \arrow[d, "\Delta"] & & \\
Y \arrow[urr, "g"] \arrow[dr, "\Gamma_g"'] & & X\times_S X \arrow[rr] \arrow[dd] & & X \arrow[dl] \arrow[dd, "f"] \\
& Y\times_S X \arrow[ur, "g\times id"] \arrow[rr] \arrow[dd] & & X \arrow[dd, "f"] & \\
& & X \arrow[rr, "f"] & & S \arrow[dl] \\
& Y \arrow[ur, "g"] \arrow[rr, "fg"] & & S &
\end{tikzcd}$$

So Γ_g is the pullback of Δ, and since Δ is an open immersion, so is Γ_g, and thus is étale.

∎

Exercise 11: Consider morphisms $Y \xrightarrow{g} X \xrightarrow{f} S$ with fg a closed immersion and f is separated. Then g is closed immersion. (Use the above technique).

Smooth, Ramified, and Étale Functors

Def: A contravariant functor $F: Sch/_X \longrightarrow Sets$ is formally smooth (lisse), resp. formally unramified (net), formally étale if for all X-schemes X' and any closed X-subscheme $X_0' \hookrightarrow X'$ defined by a nilpotent ideal, $F(X_0') \longleftarrow F(X')$ is surj, inj, or bijective resp.

A scheme $Y/_X$ is the above if $h_Y = Hom_X(-, Y)$ has the same property. If $Y/_X$ is LoFT, then $Y \to X$ is smooth, unramified, étale, if h_Y has the same formal properties.

Thm: These notions coincide with the "classical" ones.

Thm (3.23 in EC): Let $X_0 \subset X$ be a closed subscheme defined by a nilpotent ideal. Then there is an equivalence of categories $Et/X_0 \xrightarrow{\sim} Et/X$.

Exercise 12 (3): Prove this theorem (understand & use 3.10 – 3.12).

Project 4: Prove Prop. 3.24 in EC.