On Removing a Vertex from the Assignment Polytope

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ABSTRACT

Necessary and sufficient conditions are given for a doubly stochastic matrix D to be expressible as a convex combination of permutation matrices distinct from the identity, thus solving a problem posed by L. Mirsky [12].

INTRODUCTION

A square matrix is *doubly stochastic* if its entries are nonnegative real numbers and if the sum of the entries in each row and in each column is equal to 1. It is a well-known theorem of G. Birkhoff [3] and J. von Neumann [15] that any such matrix may be expressed as a convex combination of the permutation matrices. Geometrically this means that the set Δ_n of all doubly stochastic matrices of size $n \times n$ forms a bounded convex polyhedron in n^2-dimensional Euclidean space having the $n!$ permutation matrices as its only vertices. Because of its intimate connection with the famous "optimal assignment problem," the convex set Δ_n has been called the *assignment polytope* [2, 4, 5].

At the end of his 1963 survey paper on doubly stochastic matrices [12], L. Mirsky posed, among others, the following problem: What is the convex hull of all $n \times n$ permutation matrices other than the unit matrix? From what is known about the geometry of convex sets generally, and about the convex hull of the $n \times n$ permutation matrices in particular, one expects that a full answer to Mirsky's question should entail the explicit display, for each $n > 1$, of a certain minimal list of linear inequalities in n^2 variables such that, if

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these inequalities are satisfied by the elements of a doubly stochastic matrix D, then this will suffice to insure that a barycentric expression for D in terms of permutation matrices can be achieved without involving the identity matrix. This expectation recedes, however, with the realization that a knowledge of such inequalities does not necessarily offer a practical way of deciding, for a given matrix D, whether it lies in the convex hull of the nonidentity permutation matrices or not, because the sought-for inequalities may turn out to be so numerous that the task of checking them all would not ordinarily be feasible; indeed, this appears to be the actual situation. Therefore we think it appropriate to answer Mirsky's question in a somewhat different way, emphasizing our desire for a practical criterion by which matrices of moderate size can be recognized as belonging or not belonging to the aforesaid polyhedron.

Accordingly, what we show in this note is that one can always determine, for any given doubly stochastic matrix D, whether D is expressible as a convex combination of the nonidentity permutation matrices or not, by computing the solution to a certain linear-programming problem, in which the number of variables and the number of constraints are polynomial functions of the order of D. This type of criterion embodies an implicit characterization of the linear inequalities referred to earlier, though in general it does not appear to yield their explicit display. A surprising feature of the criterion is the new connection it establishes between the classical assignment problem and another much-studied branch of combinatorics, namely, the theory of (generalized) tournaments (see [14]).

TRANSITIVE TOURNAMENTS AND DIRECTED CYCLES

Let E be the set of all directed edges $\overrightarrow{uv}$ in a complete finite directed graph G without loops or multiple edges. A subset C of E is called an *elementary cycle* in G if there is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ in G such that

$$C = \{ \overrightarrow{v_1 v_2}, \overrightarrow{v_2 v_3}, \ldots, \overrightarrow{v_{k-1} v_k}, \overrightarrow{v_k v_1} \}.$$

The number k is the *length* of the cycle C. A subset T of E is called a *tournament* in G if, for every pair u, v of distinct vertices in G, exactly one of the edges $\overrightarrow{uv}$ or $\overrightarrow{vu}$ belongs to T. The tournament is *transitive* if it contains no elementary cycles. The following observation is of interest in connection with the combinatorial theory of blocking systems introduced by Edmonds and Fulkerson [8, 9]:

THEOREM 1. *For any subset S of E, either S includes a transitive tournament, or $E-S$ includes an elementary cycle, but not both.*

Proof. If $E-S$ does not include any elementary cycles, then $E-S$ is included in a maximal acyclic edge set R whose complement $E-R$ will be a transitive tournament that is included in S. On the other hand, if S includes a transitive tournament T, then $E-T$ will be an acyclic edge set which includes $E-S$, and hence $E-S$ cannot include any elementary cycles. ∎

As an application, we note that Theorem 1 may be used to give a new proof of a result on "sum-symmetric" matrices (due to S. N. Afriat [1]) which will be used in the sequel. An $n\times n$ matrix A with nonnegative entries a_{ij} is called *sum-symmetric* if, for each $k=1,2,\dots,n$, we have $\sum_{i=1}^{n}a_{ik}=\sum_{j=1}^{n}a_{kj}$. The simplest examples of sum-symmetric matrices are the *cycle matrices* (i.e., zero-one matrices $Z=(z_{ij})$ of size $n\times n$ for which the set of pairs $\{(i,j):z_{ij}=1\}$ forms an elementary cycle in the complete digraph G having the integers $1,2,\dots,n$ as its vertices), and the *loop matrices* (i.e., zero-one matrices with a single non-zero entry which is located on the main diagonal). Geometrically the set of all sum-symmetric $n\times n$ matrices may be viewed as a polyhedral convex cone, whose edges, or "extreme rays," are identified in the following theorem.

THEOREM 2 (Afriat [1]). *In order for a nonnegative square matrix A to be sum-symmetric, it is both necessary and sufficient that A can be written as a linear combination*

$$A=\theta_1Z_1+\theta_2Z_2+\cdots+\theta_mZ_m,$$

in which each Z_i is a cycle matrix or loop matrix, and each θ_i is a nonnegative scalar.

Proof (sketch). Sufficiency is trivial. Necessity is by induction on the number of positive entries in the given matrix A. If there exists any $n\times n$ cycle matrix $A=(z_{ij})$ such that $z_{ij}=1$ implies $a_{ij}>0$, then let θ be the positive scalar defined by

$$\theta=\min\{a_{ij}:z_{ij}=1\}$$

and apply the inductive hypothesis to the matrix $A-\theta Z$, which is sum-symmetric and has fewer positive entries than A. Otherwise, if no such cycle matrix Z exists, then Theorem 1 implies that the set of pairs $\{(i,j): i\neq j$ and $z_{ij}=0\}$ includes a transitive tournament; since A is sum-symmetric, it follows that A is a diagonal matrix, and hence A can be written as a nonnegative linear combination of $n\times n$ loop-matrices. ∎

As a corollary to the preceding theorem, we note that, if all entries in the given sum-symmetric matrix A are integers, then the scalars which arise in the resulting nonnegative linear combination likewise may be taken as integers. By specializing this observation to zero-one matrices, we obtain a matrix version of the well-known graph-theoretic result (reminiscent of Euler's criterion for traversability in undirected graphs) that a nontrivial digraph G is a union of edge-disjoint cycles if, and only if, at each node in G, the indegree and outdegree are equal.

GENERALIZED TRANSITIVE TOURNAMENTS

An $n\times n$ matrix $T=(t_{ij})$ of zeros and ones is called a *tournament matrix* if the set of pairs $\{(i,j): t_{ij}=1\}$ is a tournament in the complete digraph G defined on the vertices $1,2,\ldots,n$. It is easy to see that the set of all $n\times n$ tournament matrices is identical with the set of *integer* solutions to the following system of linear relations (in which distinct subscripts represent distinct indices):

$$t_{ij} \geqslant 0, \tag{3.1}$$

$$t_{ii} = 0, \tag{3.2}$$

$$t_{ij}+t_{ji} = 1. \tag{3.3}$$

Those tournaments which are transitive are represented by matrices which satisfy also the following further relations:

$$t_{ij}+t_{jk}+t_{ki} \geqslant 1. \tag{3.4}$$

J. W. Moon in [13] has broadened the concept of a tournament by referring to arbitrary real solutions of the system (3.1)–(3.3) as *generalized tournaments*. Following Moon, we shall refer to arbitrary real solutions of the system (3.1)–(3.4) as *generalized transitive tournaments*. It follows that the set $\mathcal{T}_n$ of all generalized transitive $n\times n$ tournaments forms a bounded convex polyhedron in n^2-dimensional Euclidean space. Among its extreme points are the $n!$ zero-one matrices which represent ordinary tournaments that are transitive, such as the 6×6 matrix displayed on the left below:

$$\begin{matrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & \frac{1}{2} & 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \end{matrix}$$

In general, though, the polyhedron $\mathfrak{T}_n$ will have other extreme points as well; for example, the 6×6 matrix shown above on the right is also an extreme point of the polyhedron $\mathfrak{T}_6$. (We are indebted to R. A. Brualdi for the idea which led to this example.) The difficulty in characterizing all extreme points of the polyhedron $\mathfrak{T}_n$ seems to be the main obstacle to a complete listing of facets for the polyhedron which is generated by the nonidentity permutations, in view of the following theorem.

THEOREM 3. *In order for an $n\times n$ matrix $X=(x_{ij})$ to be expressible as a convex combination of permutation matrices different from the identity, it is necessary and sufficient that X be doubly stochastic and satisfy the inequalities*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}x_{ij} \geqslant 1 \qquad (*)$$

for every generalized transitive $n\times n$ tournament $T=(t_{ij})$.

Our proof of Theorem 3 requires several preliminary results, to be established in the section which follows. Observe, however, that Theorem 3 yields at once the following corollary, which offers a procedure for deciding whether a doubly stochastic matrix D can be expressed as a convex combination of nonidentity permutations.

COROLLARY. *Let $D=(d_{ij})$ be any doubly stochastic $n\times n$ matrix, and consider the linear programming problem*

$$\text{Minimize} \quad \alpha = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}t_{ij}$$

in which $T=(t_{ij})$ *is constrained by the relations* (3.1)–(3.4). *Then D lies in the convex hull of the non-identity permutation matrices if, and only if,* $\min \alpha \geqslant 1$.

In view of this corollary, it will always be possible to determine whether or not the doubly stochastic matrix D has a barycentric expression in terms of nonidentity permutation matrices, by using the simplex method (see [7]) to solve a linear programming problem with n^2 nonnegative variables and

$$\binom{n}{1}+\binom{n}{2}+\binom{n}{3}=\frac{n(n^2+5)}{6}$$

linear constraints.

The remainder of this paper is devoted to proving Theorem 3. (A generalization of this theorem, announced in [6], will be the subject of a subsequent paper by the author: "On removing prescribed vertices from a convex polyhedron.")

PRELIMINARY RESULTS

If A and B are any two $n\times n$ matrices, we will denote by (A, B) their Euclidean inner product:

$$(A,B)=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}b_{ij}.$$

We will write the matrix inequality $A\leqslant B$ in case the relations $a_{ij}\leqslant b_{ij}$ hold for all indices $i,j=1,2,\ldots,n$. With this notation we can conveniently state and prove the following results.

THEOREM 4. *Given any generalized transitive* $n\times n$ *tournament* $T=(t_{ij})$, *and any index* $r=1,2,\ldots,n$, *there exists a unique generalized transitive* $n\times n$ *tournament* $T'=(t'_{ij})$ *such that* $t'_{rj}=0$ *for each* $j=1,2,\ldots,n$, *and such that the equation* $(T,S)=(T',S)$ *holds for every sum-symmetric* $n\times n$ *matrix* S.

Proof. Let T' be the $n\times n$ matrix defined, for all i, j, by the equation $t'_{ij}=t_{ri}+t_{ij}-t_{rj}$. Let us show that this T' has the required properties. Note that $t'_{ii}=t_{ii}=0$ holds for each index i, and that $t'_{rj}=t_{rr}=0$ holds for each index j. If i, j are distinct indices, then $t'_{ij}+t'_{ji}=t_{ij}+t_{ji}=1$. Also, when i, j are distinct indices different from r, then we get $t'_{ij}+1=t'_{ij}+(t_{rj}+t_{jr})=t_{ri}+t_{ij}+$

$t_{ir} \geqslant 1$, from which we infer that $t'_{ij} \geqslant 0$. Since $t'_{ir} = 1$ if $i \neq r$, we conclude that T' is nonnegative in all its entries. Thus T' is a generalized tournament. To deduce the equation $(T,S) = (T',S)$ when S is an arbitrary sum-symmetric matrix, which also implies that T' is transitive, it is sufficient in view of Theorem 2 and the linearity of the inner product if we show that $(T,Z) = (T',Z)$ holds whenever Z is a cycle matrix or loop matrix. This has just been noted for loops and for cycles of length 2, but indeed for arbitrary cycles it follows directly from the definition of the elements t'_{ij} and the rules of addition and subtraction. Finally, the uniqueness of T' may be seen by observing that the defining equations $t'_{ij} = t_{ri} + t_{ij} - t_{rj}$ can be obtained as algebraic combinations of the equations $(T,S) = (T',S)$ and the equations $t'_{rj} = 0$. This completes the argument. ∎

We remark that Theorem 4 is useful in identifying redundancies among the constraints (*) in Theorem 3. [For example, if T denotes the 6×6 matrix with entries 0, $\frac{1}{2}$, 1 which was displayed in the preceding section, then Theorem 4 informs us that, for all X in Δ_6, the two constraints $(T,X) \geqslant 1$ and $(T',X) \geqslant 1$ are equivalent. Yet, whatever the choice for r, the matrix T' of Theorem 4 will turn out to be the midpoint of an edge of the convex polyhedron $\mathfrak{T}_6$. The two endpoints of this edge will be zero-one tournament matrices that represent suitable cyclic permutations of the orderings (132546) and (142635). Consequently, since T' will not be an extreme point of $\mathfrak{T}_6$, the constraint $(T',X) \geqslant 1$, and therefore also the constraint $(T,X) \geqslant 1$, will be redundant in Theorem 3.] Besides this type of application, Theorem 4 also has the following corollary, which will play a key role in our proof of Theorem 3.

COROLLARY. *For any generalized transitive $n \times n$ tournament $T = (t_{ij})$, there exists a generalized transitive $n \times n$ tournament $T^* = (t^*_{ij})$ such that $(T,S) = (T^*,S)$ holds for every sum-symmetric $n \times n$ matrix S, and such that $t^*_{ij} > 0$ whenever $i \neq j$.*

Proof. For each $r = 1,2,\ldots,n$, let $T_r = (t^r_{ij})$ denote the unique generalized transitive $n \times n$ tournament such that $t^r_{rj} = 0$ for all j, and such that $(T,S) = (T_r,S)$ holds for every $n \times n$ sum-symmetric matrix S. (The existence of these matrices $T_1, T_2, \ldots, T_n$ is assured by Theorem 4.) Note that $t^r_{ir} = 1$ if $i \neq r$. Now let T^* be the average of these n matrices, i.e.:

$$T^* = (1/n)(T_1 + T_2 + \cdots + T_n).$$

Since the set $\mathfrak{T}_n$ of all generalized transitive $n \times n$ tournaments is convex, the matrix T^* is a generalized transitive $n \times n$ tournament. For any sum-symmetric $n \times n$ matrix S, the equation $(T,S)=(T^*,S)$ now follows by the linearity of the inner product:

$$(T^*,S) = \frac{1}{n}(T_1,S) + \frac{1}{n}(T_2,S) + \cdots + \frac{1}{n}(T_n,S) = (T,S).$$

Finally, if $i \neq j$, we have $t^*_{ij} = (1/n)(t^1_{ij} + t^2_{ij} + \cdots + t^n_{ij}) \geq (1/n)(t^j_{ij}) = 1/n > 0$, which completes the argument. ■

The Edmonds-Fulkerson concept of a blocking system, mentioned in connection with Theorem 1, has been generalized by Fulkerson in [10]. If $\mathfrak{C}_n$ denotes the set of all $n \times n$ cycle matrices, then in Fulkerson's terminology the set $\mathfrak{L}_n$, consisting of all nonnegative $n \times n$ matrices X such that $(Z,X) \geq 1$ holds for each Z in $\mathfrak{C}_n$, is the *blocking polyhedron* for the set $\mathfrak{C}_n$. The following decomposition theorem implies that this unbounded polyhedron $\mathfrak{L}_n$ has exactly the same extreme points as does the bounded polyhedron $\mathfrak{T}_n$ of all generalized transitive $n \times n$ tournaments.

THEOREM 5. *If $X = (x_{ij})$ is an arbitrary nonnegative $n \times n$ matrix, then X can be decomposed as $X = T + N$, where the matrix N is nonnegative and the matrix T is a generalized transitive tournament, if and only if X satisfies the inequality $(Z,X) \geq 1$ for each $n \times n$ cycle matrix Z.*

Proof. First we observe that the relation $(Z,T) \geq 1$ must hold for any generalized transitive $n \times n$ tournament T and any $n \times n$ cycle matrix Z. This is clear when Z represents an elementary cycle of length 2, since then we have $(Z,T) = t_{ij} + t_{ji} = 1$ for any generalized tournament T, by definition [condition (3.3)]. Similarly, if Z represents an elementary cycle of length 3, we get the inequality $(Z,T) = t_{ij} + t_{jk} + t_{ki} \geq 1$ as part of the definition of a generalized transitive tournament [condition (3.4)]. Finally, the general relation $(Z,T) \geq 1$ can be obtained using mathematical induction on the length k of the elementary cycle $\{(i,j): z_{ij} = 1\}$ which Z represents, since, for example, if $k > 3$, then an inequality of the form

$$t_{12} + t_{23} + t_{34} + \cdots + t_{k1} \geq 1$$

can be deduced by adding the two shorter inequalities

$$t_{12} + t_{23} + t_{31} \geq 1,$$

$$t_{13} + t_{34} + \cdots + t_{k1} \geq 1,$$

and then subtracting the equation $t_{13}+t_{31}=1$. Accordingly, for $X=T+N$, we get $(Z,X)=(Z,T)+(Z,N)\geqslant 1+0=1$, which confirms the necessity of the inequalities in the theorem.

Conversely, to demonstrate that these inequalities also are sufficient, we first note that the polyhedron $\mathfrak{L}_n$, which consists of all nonnegative $n\times n$ matrices X that satisfy these inequalities, is the vector sum of the convex hull of its set of extreme points and the nonnegative orthant of n^2-dimensional space (see [10]), and therefore it will suffice if we show that every extreme point of $\mathfrak{L}_n$ is a generalized transitive tournament. Notice that a matrix $X=(x_{ij})$ in $\mathfrak{L}_n$ cannot be extreme if $x_{ii}>0$ holds for some index i, since such an entry clearly can be either increased or decreased by a small amount without violating any of the linear constraints which define $\mathfrak{L}_n$. For the same reason, a matrix X in $\mathfrak{L}_n$ cannot be extreme if $x_{ij}>1$ holds for some choice of indices i, j. Thus, since the inequalities $x_{ij}+x_{ji}\geqslant 1$ and $x_{ij}+x_{jk}+x_{ki}\geqslant 1$, where i, j, k are distinct, are included among the constraints which define $\mathfrak{L}_n$, it remains only to show that a matrix X in $\mathfrak{L}_n$ cannot be extreme unless $x_{ij}+x_{ji}\leqslant 1$ holds for any choice of $i\neq j$. For this we argue by contradiction. Let X be an extreme point of $\mathfrak{L}_n$, and suppose $x_{uv}+x_{vu}>1$ holds for some pair of distinct indices u, v. We have already noted that all elements x_{ij} of X must satisfy $0\leqslant x_{ij}\leqslant 1$, so it follows that $x_{uv}>0$ and $x_{vu}>0$. Now we claim that at least one of these entries x_{uv} or x_{vu} can be decreased, as well as increased, by a small amount without violating any of the constraints which define $\mathfrak{L}_n$. For if this were not so, there would have to exist a pair of $n\times n$ cycle-matrices $Z_1=(z_{ij}^1)$ and $Z_2=(z_{ij}^2)$, representing elementary cycles of lengths larger than 2, such that $z_{uv}^1=z_{vu}^2=1$ and $(Z_1,X)=(Z_2,X)=1$. To see that this is impossible, let $Z_0=(z_{ij}^0)$ denote the $n\times n$ cycle matrix defined by $z_{uv}^0=z_{vu}^0=1$ and otherwise $z_{ij}^0=0$. Then the matrix S defined by $S=Z_1+Z_2-Z_0$ satisfies

$$(S,X)=(Z_1,X)+(Z_2,X)-(Z_0,X)<1+1-1=1,$$

since our hypothesis was that $(Z_0,X)>1$. Clearly, however, the matrix S is nonnegative and sum-symmetric, with integral entries and zero trace, and thus it follows from Theorem 2 that there exists an $n\times n$ cycle matrix Z_3 such that $Z_3\leqslant S$. Using the fact that X lies in $\mathfrak{L}_n$, we get the relation

$$1\leqslant (Z_3,X)\leqslant (S,X),$$

which contradicts the inequality $(S,X)<1$ obtained above. This shows that each extreme point of $\mathfrak{L}_n$ is a generalized transitive tournament, and completes the proof of Theorem 5. ∎

Proof of Theorem 3. Using the results established in the preceding sections, as well as standard facts from linear programming (see [7] or [11]), we prove in this section that a doubly stochastic $n\times n$ matrix $D=(d_{ij})$ lies in the convex hull of the nonidentity permutations if and only if D satisfies the inequality $(T,D)\geqslant 1$ for every generalized transitive $n\times n$ tournament $T=(t_{ij})$. (This answers the question of L. Mirsky which was quoted in the introduction.)

To show the necessity of these inequalities [labeled (*) in the statement of Theorem 3], it will suffice to prove that they hold for every nonidentity permutation matrix. But since any permutation matrix P is sum-symmetric and consists entirely of zeros and ones, it follows from Theorem 2 that P can be written as $P=Z_1+Z_2+\cdots+Z_m$, where each Z_i is either a cycle matrix or a loop matrix; furthermore, if P is not the identity, then at least one of the Z_i's, say Z_1, must be a cycle matrix. Now if T is any generalized transitive $n\times n$ tournament, then from Theorem 5 we get $(T,Z_1)\geqslant 1$. Since $P\geqslant Z_1$, the inequality $(T,P)\geqslant(T,Z_1)\geqslant 1$ follows. Thus the conditions (*) of Theorem 3 indeed are necessary.

Conversely, suppose $D=(d_{ij})$ is a doubly stochastic $n\times n$ matrix which satisfies $(D,T)\geqslant 1$ for every generalized transitive $n\times n$ tournament $T=(t_{ij})$, and consider the following linear programming problem, in which we seek to minimize the value of the inner product (D,X) as X varies over all points of the blocking polyhedron $\mathfrak{L}_n$ for the set $\mathfrak{C}_n$ of all the $n\times n$ cycle matrices:

$$\text{Minimize}\quad \alpha=\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}x_{ij}$$

constrained by

$$x_{ij}\geqslant 0\qquad \text{for all}\quad i,j \text{ in } \{1,2,\ldots,n\},$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} z_{ij}x_{ij}\geqslant 1\qquad \text{for all}\quad Z \text{ in } \mathfrak{C}_n.$$

Theorem 5 informs us that $\min\alpha$ will be attained at some point X which is a generalized transitive tournament, and therefore, in view of our hypothesis on D, we will have $\min\alpha\geqslant 1$. We distinguish two cases.

First, suppose $\min\alpha=1$. In this case, we consider the following maximum problem (which is "dual" to the minimum problem shown above):

$$\text{Maximize}\quad \beta=\sum_{Z\in\mathfrak{C}_n} yZ$$

constrained by

$$y_Z \geqslant 0 \qquad \text{for all} \quad Z \text{ in } \mathfrak{C}_n$$

$$\sum_{Z \in \mathfrak{C}_n} z_{ij} y_Z \leqslant d_{ij} \qquad \text{for all} \quad i,j \text{ in } \{1,2,\ldots,n\}.$$

By the fundamental "duality principle" of linear programming, we are guaranteed that $\max\beta = \min\alpha = 1$. Furthermore, the principle of "complementary slackness" implies that, if $X = (x_{ij})$ is any optimal solution to the minimum problem, and $Y = (y_Z : Z \in \mathfrak{C}_n)$ is any optimal solution to the maximum problem, then we must have

$$x_{ij} > 0 \quad \text{implies} \quad \sum_{Z \in \mathfrak{C}_n} z_{ij} y_Z = d_{ij}.$$

But the Corollary to Theorem 4 informs us that an optimal X may be found with the property that $x_{ij} > 0$ whenever $i \neq j$, which implies that the equations

$$d_{ij} = \sum_{Z \in \mathfrak{C}_n} z_{ij} y_Z$$

must hold for all the "off-diagonal" elements d_{ij} in the matrix D. Now, for each $n \times n$ cycle matrix Z in $\mathfrak{C}_n$, let Z^* denote the unique $n \times n$ permutation matrix such that $z_{ij} = z_i^*$ holds for all $i \neq j$. Note that none of the Z^*'s is equal to the identity matrix. Since we have $1 = \max\beta = \sum_{Z \in \mathfrak{C}_n} y_Z$, and $y_Z \geqslant 0$ for all Z in $\mathfrak{C}_n$, the equations displayed above allow us to deduce the following matrix equality:

$$D = \sum_{Z \in \mathfrak{C}_n} y_Z Z^*,$$

since the matrix on the right-hand side is doubly stochastic (in view of the convexity of the set Δ_n), and because two doubly stochastic $n \times n$ matrices are clearly identical if all of their corresponding "off-diagonal" entries are the same. Thus we have achieved an expression for the matrix D as a convex combination of the nonidentity permutation matrices Z^*, as required.

Next, suppose $\min\alpha > 1$. If any diagonal element d_{ii} is equal to zero, then it follows directly from the Birkhoff–von Neumann theorem cited in the introduction that D can be written as a convex combination of nonidentity permutation matrices. So we may assume here, without loss of generality, that $d_{ii}>0$ holds for all $i=1,2,\dots,n$. In this case, let $w=\min\alpha$, and consider the $n\times n$ matrix $D^*=(d_{ij}^*)$ defined by the rule

$$d_{ij}^* = \begin{cases} (1/w)d_{ij}, & \text{if} \quad i\neq j \\ 1-\sum\limits_{k\neq i}(1/w)d_{ik}, & \text{if} \quad i=j. \end{cases}$$

It is readily checked that this matrix D^* has all of the following properties:

$$0\leqslant d_{ij}^*\leqslant d_{ij} \qquad \text{for all} \quad i\neq j. \tag{5.1}$$

$$0< d_{ii}\leqslant d_{ii}^* \qquad \text{for all} \quad i. \tag{5.2}$$

$$D^* \text{ is doubly stochastic.} \tag{5.3}$$

$$(D^*,T)\geqslant 1 \qquad \text{for every} \quad T \text{ in } \mathfrak{T}_n. \tag{5.4}$$

$$d_{ii}^* > d_{ii} \qquad \text{for some} \quad i \text{ in } \{1,2,\dots,n\}. \tag{5.5}$$

$$(D^*,T) = 1 \qquad \text{for some} \quad T \text{ in } \mathfrak{T}_n. \tag{5.6}$$

Now let θ be defined by $\theta=\min\{d_{ii}/d_{ii}^*:1\leqslant i\leqslant n\}$. Note that θ satisfies $0<\theta<1$, by the conditions (5.2) and (5.5). The nonnegative matrix $Q=(1-\theta)^{-1}(D-\theta D^*)$ is doubly stochastic and, by the choice of θ, has at least one diagonal element equal to zero. Hence, as noted earlier, Q lies in the convex hull of the nonidentity permutation matrices. Also, by the argument in the preceding paragraph, the matrix D^* lies in the convex hull of the nonidentity permutation matrices, by the conditions (5.3), (5.4), and (5.6). Since D can be written as $D=\theta D^*+(1-\theta)Q$, it follows that D belongs to the convex hull of the nonidentity permutation matrices, as required. This finishes our proof of Theorem 3. ■

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