AGEC 652 - Lecture 3.1

Linear Equations

Diego S. Cardoso

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Course roadmap

1. Intro to Scientific Computing
2. Numerical operations and representations
3. **Systems of equations**
    1. **Linear equations** ← You are here
    2. Nonlinear equations
4. Optimization
5. Structural estimation

*These slides are based on Miranda & Fackler (2002), Judd (1998), and course materials by Ivan Rudik.

Linear equations in Economics

(Systems of) Linear equations are very common in Economics

$$Ax = b$$

where A is a $n \times n$ matrix, b and x are n-vectors

Examples?

- Comparative statics
- General equilibrium models with linear functions
- Log-linearized models
- Steady-state distributions of discrete stochastic processes

Solving linear equations in Julia

Solving linear systems is generally very easy in programming languages

```
A = [-3 2 3; -3 2 1; 3 0 0]; b = [10; 8; -3];
x = A\b # This is an optimized division, faster than inverting A (more on that later)
```

```
## 3-element Vector{Float64}:
##  -1.0
##   2.0
##   1.0
```

- So why bother?

Linear equations: Why bother?

1. It's a *building block*: many methods decompose more complicated problems into sequences of linear problems
   - Understanding how we solve linear systems is crucial to understanding other methods
2. It uses key concepts of numerical analysis, such as iterative methods
   - Seeing these ideas in action with a familiar problem will help you understand more complex ones
3. Like any other numerical method, it is prone to limited precision issues that grow with repeated operations
   - In linear systems we can see these issues in a transparent and intuitive way

Solving linear equations

OK, so how does the computer actually solve linear equations?

Methods come in two flavors:

1. Direct methods
   - We solve it in one pass and get a solution with a finite number of operations
2. Iterative methods
   - We solve the same problem repeatedly until results converge to a solution

Solving linear equations: direct methods

Let's start with the simplest case: a *lower triangular* matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

How do we solve this?

Easy, forward substitution!

Solving linear equations: forward substitution

$$
\begin{aligned}
x_1 &= b_1 / a_{11} \\
x_2 &= (b_2 - a_{21} x_1) / a_{22} \\
x_3 &= (b_3 - a_{31} x_1 - a_{32} x_2) / a_{33} \\
&\vdots \\
x_n &= (b_3 - a_{n1} x_1 - a_{n2} x_2 - \cdots) / a_{nn}
\end{aligned}
$$

We can write a simple algorithm to solve it: $x_i = \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j \right) / a_{nn}$ for all i

What if A is *upper triangular*? We use *backward substitution* and just reverse the order

What is the complexity of this algorithm (in O notation)?

Solving linear equations: forward substitution

$x_i = \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j \right) / a_{nn}$ for all i

What is the complexity of this algorithm?

There are:

- n divisions
- $n(n-1)/2$ multiplications
- $n(n-1)/2$ additions/subtractions

Order of $n^2/2$ operations $\rightarrow O(n^2)$

LU factorization

In practice we rarely need to solve triangular systems! What if A is not triangular?

1. We decompose A into two matrices: one **U**pper triangular and one **L**ower triangular
   $\rightarrow A = LU$
   - We use Gaussian elimination for that
2. Then, we solve the problem using a combination of forward and backward substitutions
   - The system becomes $Ax = (LU)x = L\underbrace{(Ux)}_{y} = b$
   - We solve $Ly = b$ using forward substitution
   - Then $Ux = y$ using backward substitution

This works for any non-singular matrix

Gaussian elimination

This uses the fact that we can do the following operations to a linear system **without changing its solution**

1. multiply one row by a scalar and add to another row
2. swap rows

We'll use that to turn a matrix $(I A)$ into $(L U)$

LU factorization example

Let's see an example with system $Ax = b$

$$A = \begin{bmatrix} -3 & 2 & 3 \\ -3 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 10 \\ 8 \\ -3 \end{bmatrix}$$

LU factorization

We start with $I = L, A = U$ and we to make U upper triangular with Gaussian elimination

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} -3 & 2 & 3 \\ -3 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

We do `row2 = row2 - (1)*row1` and `row3 = row3 - (-1)*row1`, keeping track of these operations in the L matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & 2 & 3 \end{bmatrix}$$

LU factorization example

Seems like we are stuck in U. But we can swap rows 2 and 3 to make it upper triangular.

- *Correspondingly, we need to swap columns 2 and 3 in L*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} -3 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

Now we are ready to solve the first part of the problem: $Ly = b$

But wait, L is not lower triangular!

Not yet. But all we need is for it to be *row-permuted* lower triangular because we can easily swap rows and solve for y

LU factorization example

Solving $Ly = b$ we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ -3 \end{bmatrix}$$

is equivalent to

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \\ 8 \end{bmatrix}$$

which is easy to solve

LU factorization example

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \\ 8 \end{bmatrix}$$

We solve by forward substitution as

$$\begin{aligned} y_1 &= b_1/l_{11} = 10 \\ y_2 &= (b_2 - l_{21}y_1)/l_{22} = [-3 - (-1)(10)]/1 = 7 \\ y_3 &= (b_3 - l_{31}y_1 - l_{32}y_2)/l_{33} = [8 - (1)(10) - (0)(7)]/1 = -2 \end{aligned}$$

LU factorization example

Now that we have y, we solve $Ux = y$

$$\begin{bmatrix} -3 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -2 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ -2 \end{bmatrix}$$

We solve by backward substitution as

$$\begin{aligned} x_3 &= y_3/u_{33} = -2/(-2) = 1 \\ x_2 &= (y_2 - u_{23}y_1)/u_{22} = [7 - (3)(1)]/2 = 2 \\ x_1 &= (y_1 - u_{13}y_1 - u_{12}y_2)/u_{11} = [10 - (3)(1) - (2)(2)]/(-3) = -1 \end{aligned}$$

And done!

Why bother with this scheme?

Why not just use another method like Cramer's rule?

Speed!

- LU is less than $O(n^3)$
- Cramer's rule is $O(n! \times n)$

For a 10x10 system this can really matter:

- LU factorization: 430 long operations ($*$ and $/$)
- Matrix inversion and multiplication: 1,100 long operations
- Cramer: 40 million long operations!

Example: LU vs Cramer

Julia description of the division operator `\`:

> If A is upper or lower triangular (or diagonal), no factorization of A is required and the system is solved with either forward or backward substitution. For non-triangular square matrices, an LU factorization is used.

So we can do LU factorization to solve systems by just doing `x = A\b`. But we could write it ourselves as well

Example: LU vs Cramer

Cramer's Rule can be written as a simple loop:

```
function solve_cramer(A, b)

    dets = Vector(undef, length(b))

    for index in eachindex(b)
        B = copy(A)
        B[:, index] = b
        dets[index] = det(B)
    end

    return dets ./ det(A)

end
```

```
n = 100
A = rand(n, n)
b = rand(n)
```

Example: LU vs Cramer

Let's see the full results of the competition for a 10x10:

```
using BenchmarkTools
cramer_time = @elapsed solve_cramer(A, b);
cramer_allocation = @allocated solve_cramer(A, b);
lu_time = @elapsed A\b;
lu_allocation = @allocated A\b;

println(
"Cramer's rule solved in $cramer_time seconds and used $cramer_allocation kilobytes of memory.
LU solved in $(lu_time) seconds and used $(lu_allocation) kilobytes of memory.
LU is $(round(cramer_time/lu_time, digits = 0)) times faster
 and uses $(round(lu_allocation/cramer_allocation*100, digits = 2))%  of the memory.")
```

```
## Cramer's rule solved in 2.5417579 seconds and used 16187072 kilobytes of memory.
## LU solved in 0.0586789 seconds and used 81840 kilobytes of memory.
## LU is 43.0 times faster
##  and uses 0.51%  of the memory.
```

Example: LU vs matrix inversion

Let's see the full results of the competition for a 10x10:

```
using BenchmarkTools
invers_time = @elapsed ((A^-1)*b);
invers_allocation = @allocated ((A^-1)*b);

println(
"Matrix inversion solved in $cramer_time seconds and used $cramer_allocation kilobytes of memory.
LU solved in $(lu_time) seconds and used $(lu_allocation) kilobytes of memory.
LU is $(round(invers_time/lu_time, digits = 2)) times faster
 and uses $(round(lu_allocation/invers_allocation*100, digits = 2))%  of the memory.")
```

```
## Matrix inversion solved in 2.5417579 seconds and used 16187072 kilobytes of memory.
## LU solved in 0.0586789 seconds and used 81840 kilobytes of memory.
## LU is 0.47 times faster
##  and uses 61.46%  of the memory.
```

Other factorizations

LU is not the only direct method used to speed up linear equation solvers

- QR factorization decomposes $A = QR$, where Q is an orthogonal matrix and R is upper triangular
- Cholesky factorization can be used if A is positive definite
  - This is particularly useful in optimization, where we often get matrices like that
- There are also methods optimized for sparse matrices

Chapter 3 in Judd has a summary and references of other methods if you need them for your research in the future

Rounding error blow-up

In practice, Gaussian elimination can lead to *very inaccurate* solutions. For example:

$$\begin{bmatrix} -M^{-1} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

where M is a large positive number

Suppose we use LU factorization to solve it

$$\begin{bmatrix} -M^{-1} & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -M^{-1} & 1 \\ 1 & 1 \end{bmatrix}$$

Numerical error blow-up

Subtract $-M$ times the first row from the second to get the LU factorization

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -M^{-1} & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -M & 1 \end{bmatrix} \begin{bmatrix} -M^{-1} & 1 \\ 0 & M+1 \end{bmatrix}$$

We can get closed-form solutions by applying forward substitution:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} M/(M+1) \\ (M+2)/(M+1) \end{bmatrix}$$

Numerical error blow-up

When M is large, both variables are approximately 1

- But remember adding small numbers to big numbers causes problems numerically

If $M = 10000000000000000000$, the computer will return x_2 is equal to precisely 1

- This isn't terribly wrong

When we then perform the second step of backwards substitution, we solve for $x_1 = -M(1 - x_2) = 0$, this is **VERY** wrong

Large errors like this often occur because diagonal elements are very small

A numerical error blow-up solution

Large errors like this often occur because diagonal elements are very small

In some cases, this can be solved by **pivoting**: we swap two rows so that small numbers are off-diagonal

- Swapping rows does not change the solution and may prevent operations causing rounding errors

Most numerical linear algebra packages will do this for you (including the one embedded in Julia)

Numerical error blow-up: Julia example

```
function solve_lu(M)
    b = [1, 2]
    U = [-M^-1 1; 0 M+1]
    L = [1. 0; -M 1.]
    y = L\b
    # Round element-wise to 3 digits
    x = round.(U\y, digits = 5)
end;

true_solution(M) = round.([M/(M+1), (M+2)/(M+1)], digits = 5);
println("True solution for M=10   is approx. $(true_solution(10)), computed solution is $(solve_lu(10))");
println("True solution for M=1e10 is approx. $(true_solution(1e10)), computed solution is $(solve_lu(1e10))");
println("True solution for M=1e15 is approx. $(true_solution(1e15)), computed solution is $(solve_lu(1e15))");
println("True solution for M=1e20 is approx. $(true_solution(1e20)), computed solution is $(solve_lu(1e20))");
```

Numerical error blow-up: Julia example

```
## True solution for M=10 is approximately [0.90909, 1.09091], computed solution is [0.90909, 1.09091]

## True solution for M=1e10 is approximately [1.0, 1.0], computed solution is [1.0, 1.0]

## True solution for M=1e15 is approximately [1.0, 1.0], computed solution is [1.11022, 1.0]

## True solution for M=1e20 is approximately [1.0, 1.0], computed solution is [-0.0, 1.0]
```

```
M = 1e20;
A = [-M^-1 1; 1 1];
b = [1., 2.];
julia_solution = A\b;
println("Julia's division operator is actually pretty smart though,
        true solution for M=1e20 is $(julia_solution)")
```

```
## Julia's division operator is actually pretty smart though,
##         true solution for M=1e20 is [1.0, 1.0]
```

Ill-conditioning

A matrix A is said to be ill-conditioned if a small perturbation in b yields a large change in x

One way to measure ill-conditioning in a matrix is the elasticity of the solution with respect to b,

$$\sup_{||\delta b||>0} \frac{||\delta x||/||x||}{||\delta b||/||b||}$$

which yields the percent change in x given a percentage point change in the magnitude of b

Ill-conditioning

If this elasticity is large, then then small errors in the representation of b can lead to large errors in the computed solution x

Calculating this elasticity is computationally expensive. We approximate it by calculating the **condition number**

$$\kappa = ||A|| \cdot ||A^{-1}||$$

κ gives the least upper bound of the elasticity and is always larger than one

Ill-conditioning

Rule of thumb: *for each power of 10, a significant digit is lost in the computation of* x

```
using LinearAlgebra;
cond([1. 1.; 1. 1.0001])
```

```
## 40002.00007491187
```

```
cond([1. 1.; 1. 1.00000001])
```

```
## 4.000000065548868e8
```

```
cond([1. 1.; 1. 1.000000000001])
```

```
## 3.99940450394283e12
```

Iterative methods

Direct methods like LU factorization work well for relatively small matrices. As n gets bigger, the time and memory needed becomes prohibitive

When that happens, we use **iterative methods** instead

- They require less memory
- Adequate iterative methods can give a good answer in reasonable time

Let's start with the simplest and most intuitive iterative method

Fixed-point iteration

Main idea: *we reformulate our problem as a fixed-point problem and iterate it the mapping*

Instead of $Ax = b$, we define $G(x) \equiv Ax - b + x$

We start with an initial guess in step $k = 0$ and compute the next values using

$$x^{(k+1)} = G(x^{(k)}) = (A + I)x^{(k)} - b$$

When we find a fixed point (i.e, $x = G(x)$), we know that x solves our initial problem $Ax = b$

We don't use this method though because it is very particular: it only converges if all eigenvalues of $(A + I)$ have modulus less than one

Operator Splitting methods

In a more useful method, we can rewrite $Ax = b$ as $Qx = b + (Q - A)x$ for some square matrix Q

Rearranging, we get $x = Q^{-1}b + (I - Q^{-1}A)x$, which suggests the iterating rule

$$x^{(k+1)} = Q^{-1}b + (I - Q^{-1}A)x^{(k)}$$

It is easy to check that if $x^{(k+1)} = x^{(k)}$, then $x^{(k)}$ is a solution to $Ax = b$

- This approach can be used in nonlinear systems, too!

Operator Splitting methods

Q is called the **splitting matrix**. That's because it effectively splits A into $A = Q - P$

In practice, we choose Q so that

1. $Q^{-1}b$ and $Q^{-1}a$ are easy to compute (like when Q is diagonal or triangular)
2. $||I - Q^{-1}A|| < 1$ so we know the iteration converges

Gauss-Jacobi method

When Q is chosen to be a diagonal matrix with the same diagonal elements in A, we have the **Gauss-Jacobi method**

The intuition is simple:

- For every equation in this system, we can always write $x_i = \frac{1}{a_{ii}}[b_i - \sum_{j \neq i} a_{ij}x_j]$
- We use this equation to formulate the iteration rule

$$x_i^{(k+1)} = \frac{1}{a_{ii}}[b_i - \sum_{j \neq i} a_{ij}x_j^{(k)}]$$

Gauss-Jacobi method

Iteration rule

$$x_i^{(k+1)} = \frac{1}{a_{ii}}[b_i - \sum_{j \neq i} a_{ij} x_j^{(k)}]$$

Then, we assume initial values $x_i^{(0)}$ $\forall i$ and iterate all x_i simultaneously *until convergence*

$\Rightarrow$ we turned a "n equations with n unknowns" into repeatedly solving n equations with *1* unknown

Convergence

What do we mean by *until convergence*?

This is a parameter you have to choose

- Usually, we will set a `tolerance` value
- Then, in each iteration, we take some metric of the difference δ_x between x^{k+1} and x^k
- If $d_x < $ `tolerance`, we stop iterating and declare x^{k+1} our solution

The actual choice of `tolerance` will depend on the scale of your variables and the desired precision

Convergence

It is a good practice to set a `max_iterations` parameter to stop your code once a maximum number of iterations have run

- This avoids your code getting into an **infinite loop** if your method turn out not to converge

You can do that by incrementing a variable that counts the number of iterations and testing the condition `iteration <= max_iterations` before proceeding

- If your hit the maximum number of iteration, it's a sign your solution did not converge

Convergence

An example of how to test both the convergence and maximum iteration conditions is

```
dx = Inf # Start with a very large number
tol = 1e-6 # One example
iteration = 0 # Initialize value
max_iterations = 1000 # Set max of 1000 iterations
while (dx >= tol && iteration <= max_iterations)
  iteration = iteration + 1
  # Here you iterate your solutions and recalculate dx
end
```

Gauss-Seidel method

The Gauss-Seidel method chooses Q as an *upper triangular matrix* with the same elements in A

Here is the intuition behind it

In Gauss-Jacobi, we only use a new guess $x^{(k+1)}$ after we've computed all $x_i^{(k+1)}$

- For example, we use x_1^k to calculate $x_2^{(k+1)}$ even though we already have $x_1^{(k+1)}$

We can make faster use of information if we use our newly calculated guesses right away. That is what the *Gauss-Seidel method* does

Gauss-Seidel method

So, when we compute the new guess for x_2, we have
$x_2^{(k+1)} = (b_2 - a_{21}x_1^{(k+1)} - a_{23}x_1^{(k)} - \ldots)/a_{22}$

This give the iteration rule

$$x_i^{(k+1)} = \frac{1}{a_{ii}}[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}]$$

Unlike Gauss-Jacobi, **in Gauss-Seidel the order equations matters**

- This also makes Gauss-Seidel more flexible, because we can try different orders to get it to converge

A visual example

Tâttonement is an old concept (Walras, 1954) to describe how markets reach equilibrium by trial-and-error

1. An initial price is fixed
2. Demanded and supplied quantities are announced based on that price
3. If quantities don't match, a (Walrasian) auctioneer announces a new price and the process repeats until we reach an equilibrium
    - If there is oversupply, lower the price
    - If there is overdemand, raise the price

We can use this concept to illustrate iterative solution methods with linear demand/supply equations

A visual example

Let's consider the simple linear demand/supply problem

- Inverse demand: $p = 10 - q$
- Supply: $q = p/2 + 1$

To solve it, we form the system

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$$

From Gauss-Jacobi iteration rule

$$p^{(k+1)} = (10 - q^{(k)})/1 = 10 - q^{(k)}$$
$$q^{(k+1)} = (-2 - p^{(k)})/(-2) = 1 + p^{(k)}/2$$

A visual example: Gauss-Jacobi

But let's see how that rule comes from matrix form

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \Rightarrow Q = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow Q^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$$

So $x^{(k+1)} = Q^{-1}b + (I - Q^{-1}A)x^{(k)}$ gives

$$\begin{bmatrix} p^{(k+1)} \\ q^{(k+1)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 10 \\ -2 \end{bmatrix} + \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \right) \begin{bmatrix} p^{(k)} \\ q^{(k)} \end{bmatrix}$$

$$\begin{bmatrix} p^{(k+1)} \\ q^{(k+1)} \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} p^{(k)} \\ q^{(k)} \end{bmatrix}$$

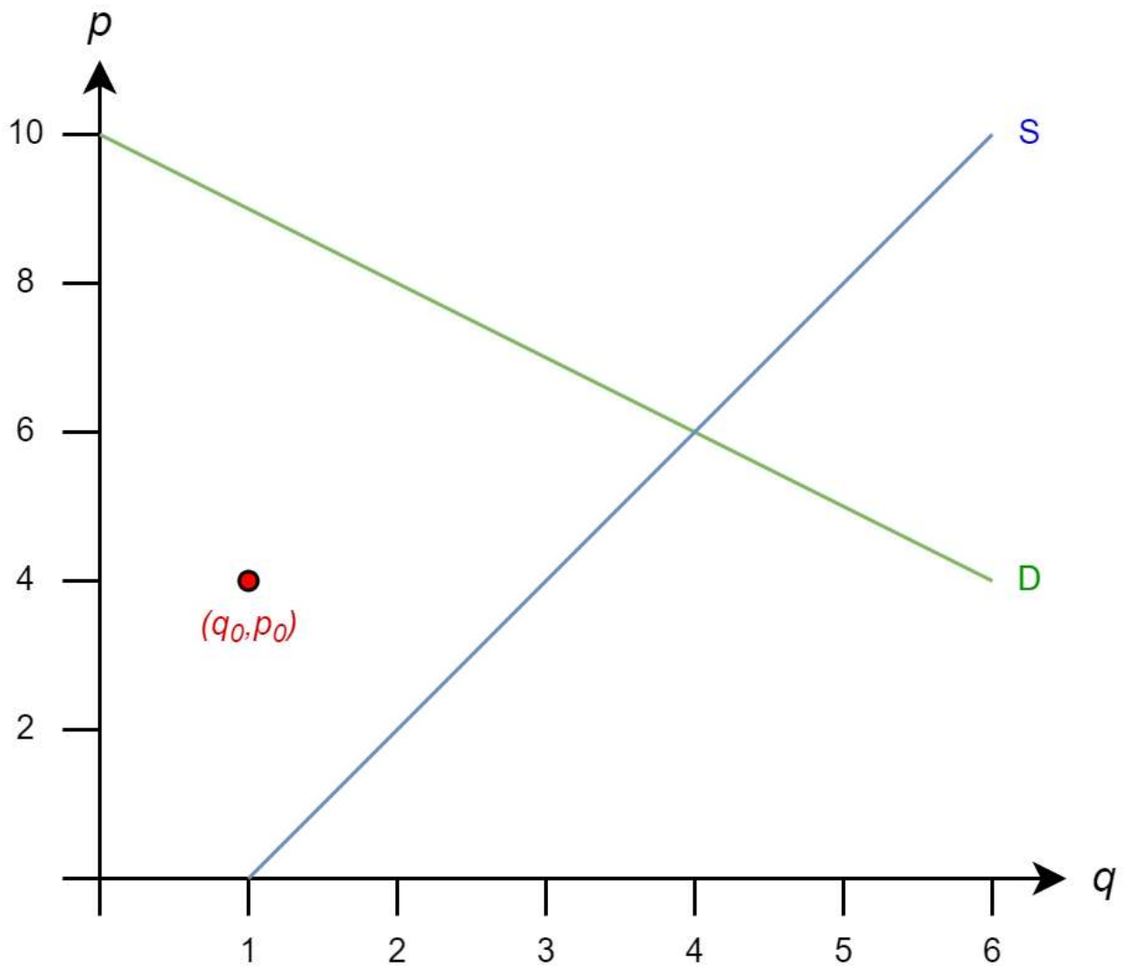
A visual example: Gauss-Jacobi

Let's start from initial guess $q_0 = 1$ and $p_0 = 4$ and see how Gauss-Jacobi proceeds

The iteration rules are

$$q^{(k+1)} = 1 + p^{(k)}/2$$

$$p^{(k+1)} = 10 - q^{(k)}$$



A visual example: Gauss-Jacobi

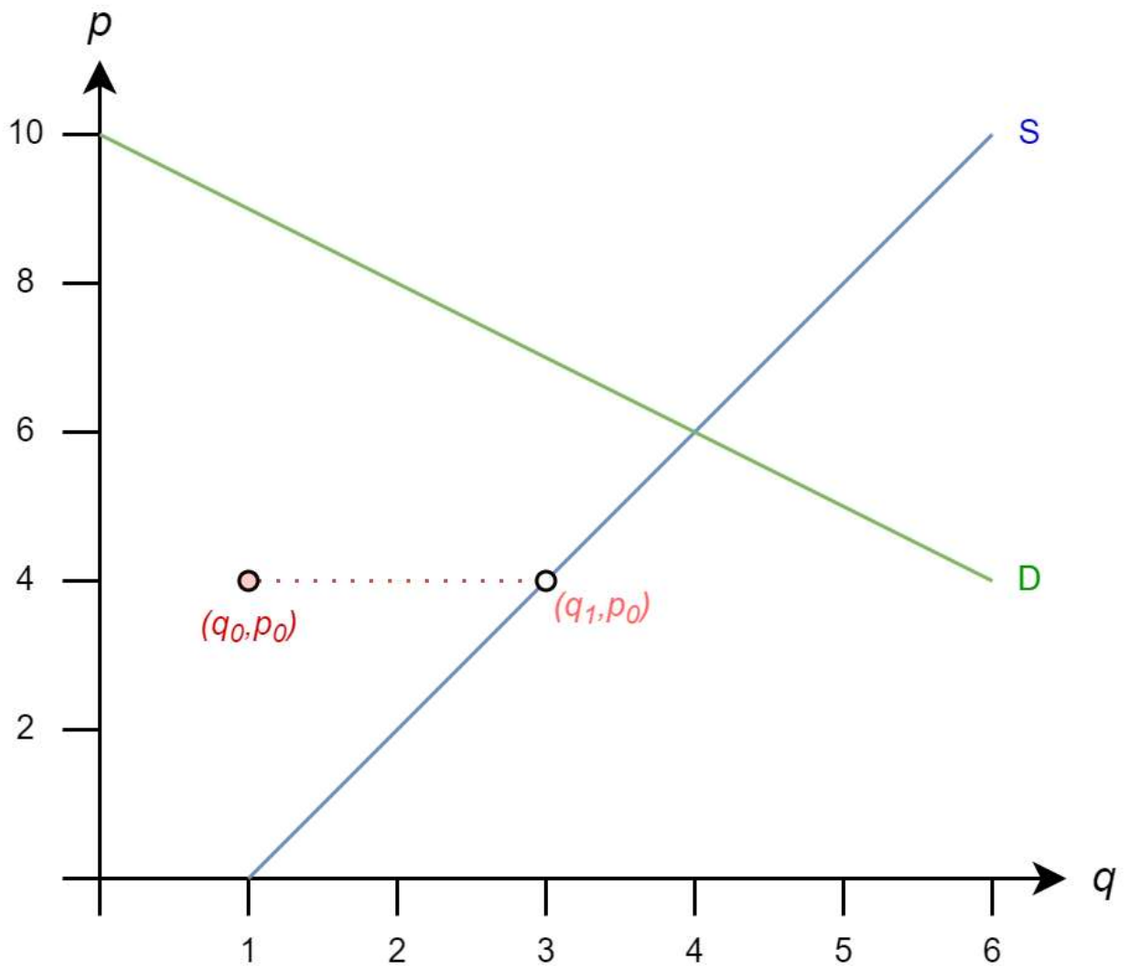
The iteration rules are

$$q^{(k+1)} = 1 + p^{(k)}/2$$

$$p^{(k+1)} = 10 - q^{(k)}$$

So

$$q_1 = 1 + (4)/2 = 3$$



A visual example: Gauss-Jacobi

The iteration rules are

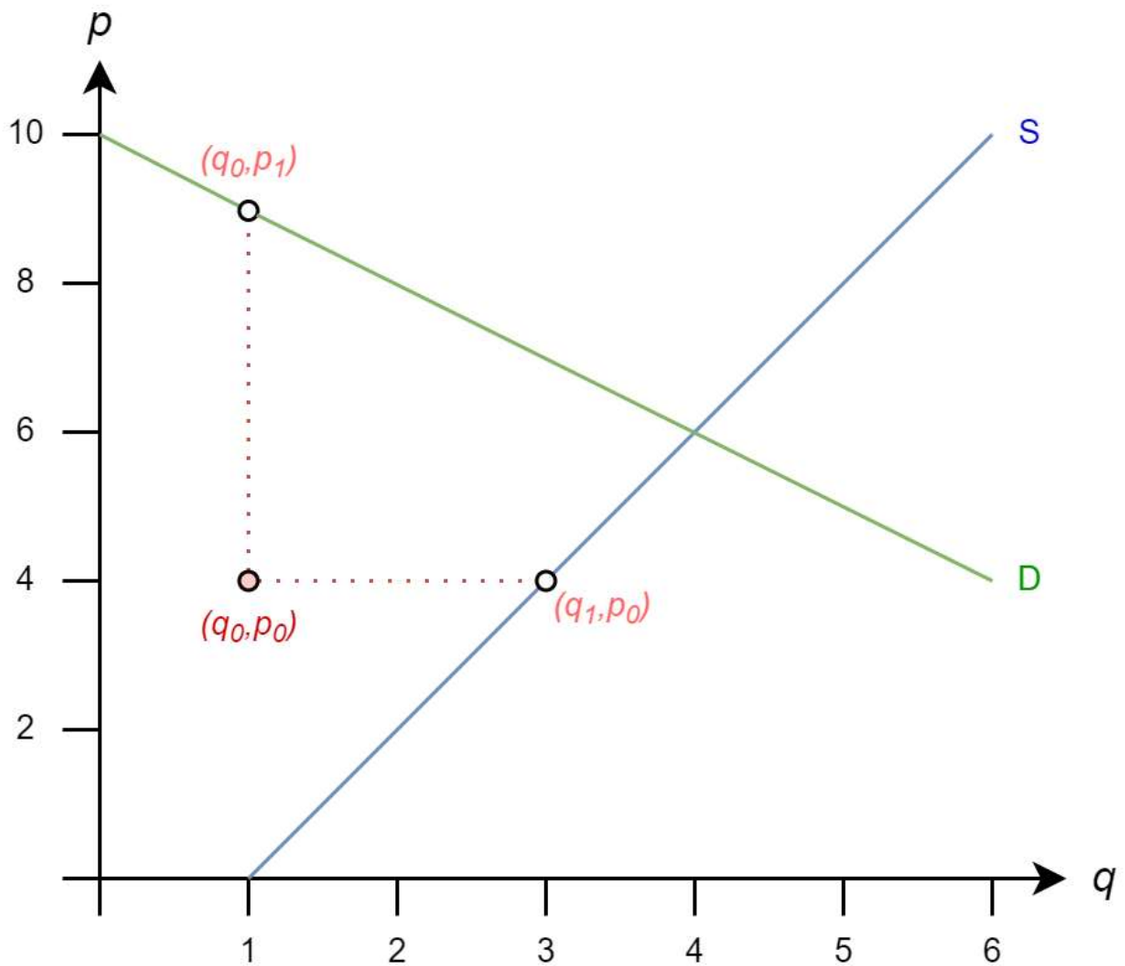
$$q^{(k+1)} = 1 + p^{(k)}/2$$

$$p^{(k+1)} = 10 - q^{(k)}$$

So

$$q_1 = 1 + (4)/2 = 3$$

$$p_1 = 10 - 1 = 9$$



A visual example: Gauss-Jacobi

The iteration rules are

$$q^{(k+1)} = 1 + p^{(k)}/2$$

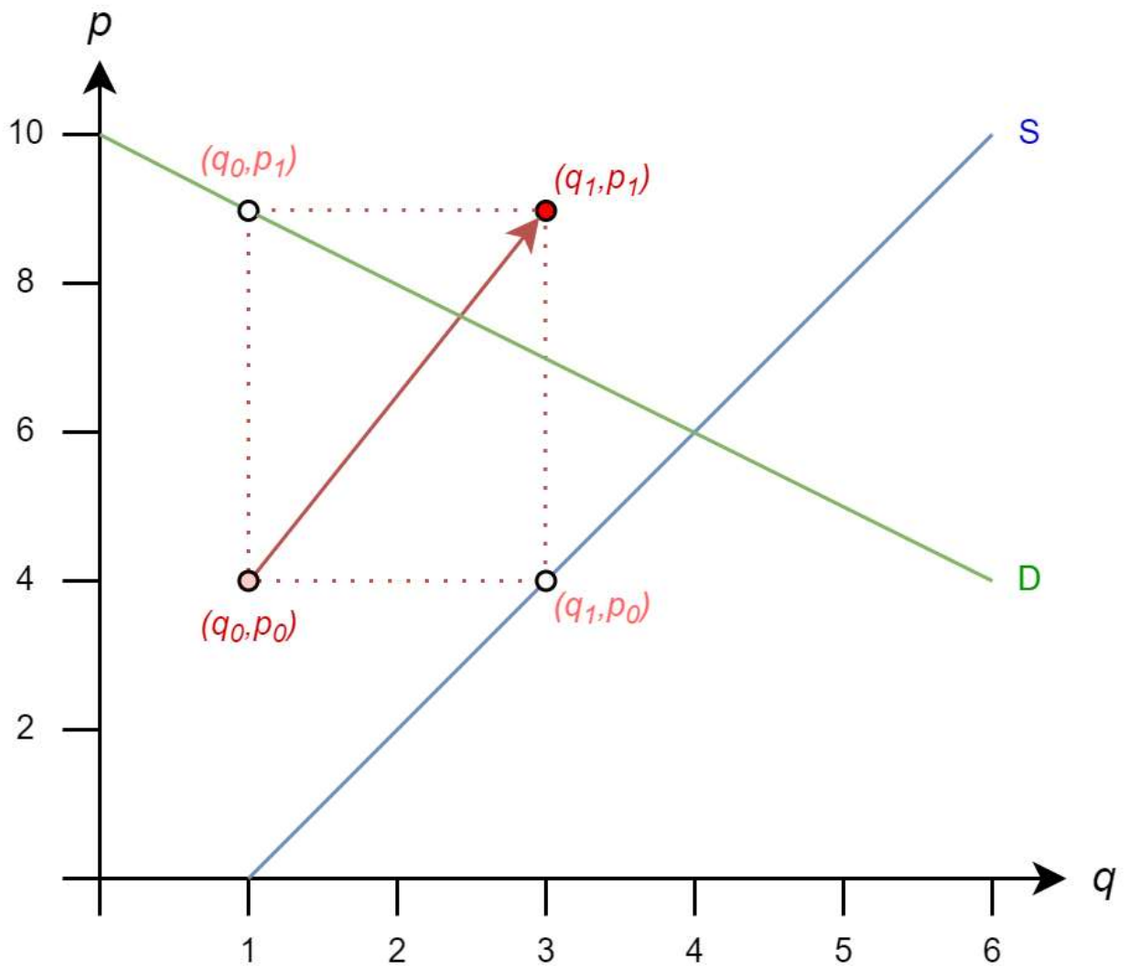
$$p^{(k+1)} = 10 - q^{(k)}$$

So

$$q_1 = 1 + (4)/2 = 3$$

$$p_1 = 10 - 1 = 9$$

And we move to $(3, 9)$



A visual example: Gauss-Jacobi

The iteration rules are

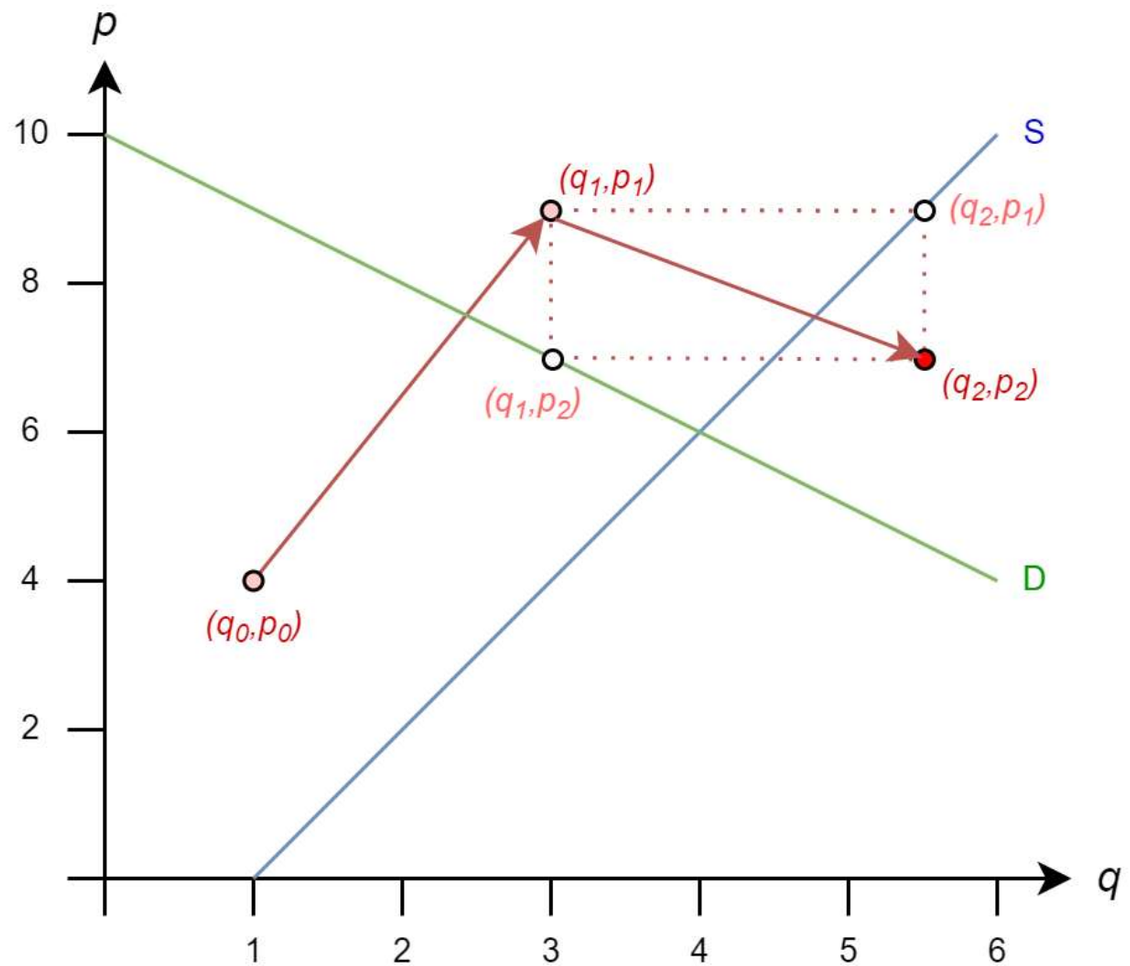
$$q^{(k+1)} = 1 + p^{(k)}/2$$

$$p^{(k+1)} = 10 - q^{(k)}$$

Next, we get $q_2 = 1 + (9)/2 = 5.5$

$p_2 = 10 - 3 = 7$

And we move to $(5.5, 7)$



A visual example: Gauss-Jacobi

The iteration rules are

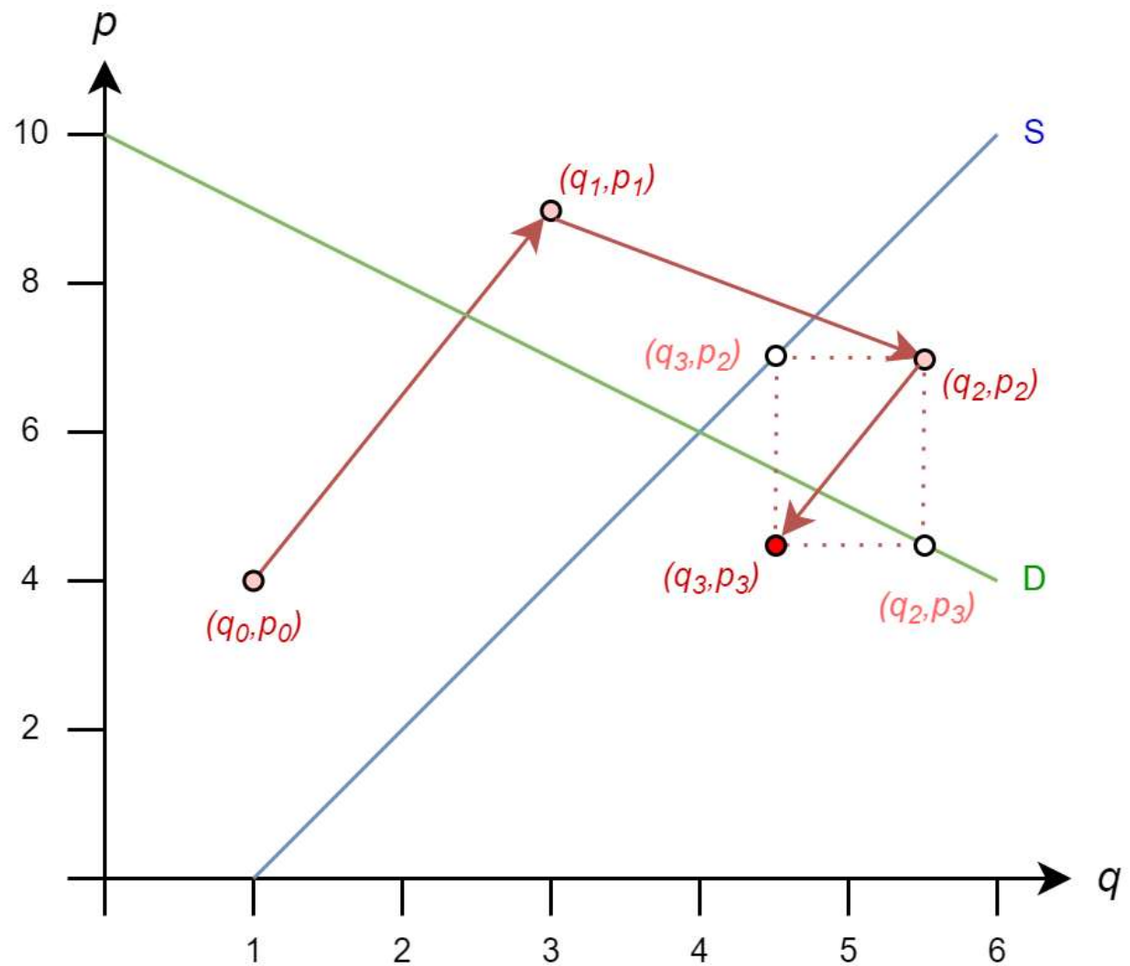
$$q^{(k+1)} = 1 + p^{(k)}/2$$

$$p^{(k+1)} = 10 - q^{(k)}$$

Then, we get $q_3 = 1 + (7)/2 = 4.5$

$p_3 = 10 - 5.5 = 4.5$

And we move to $(4.5, 4.5)$



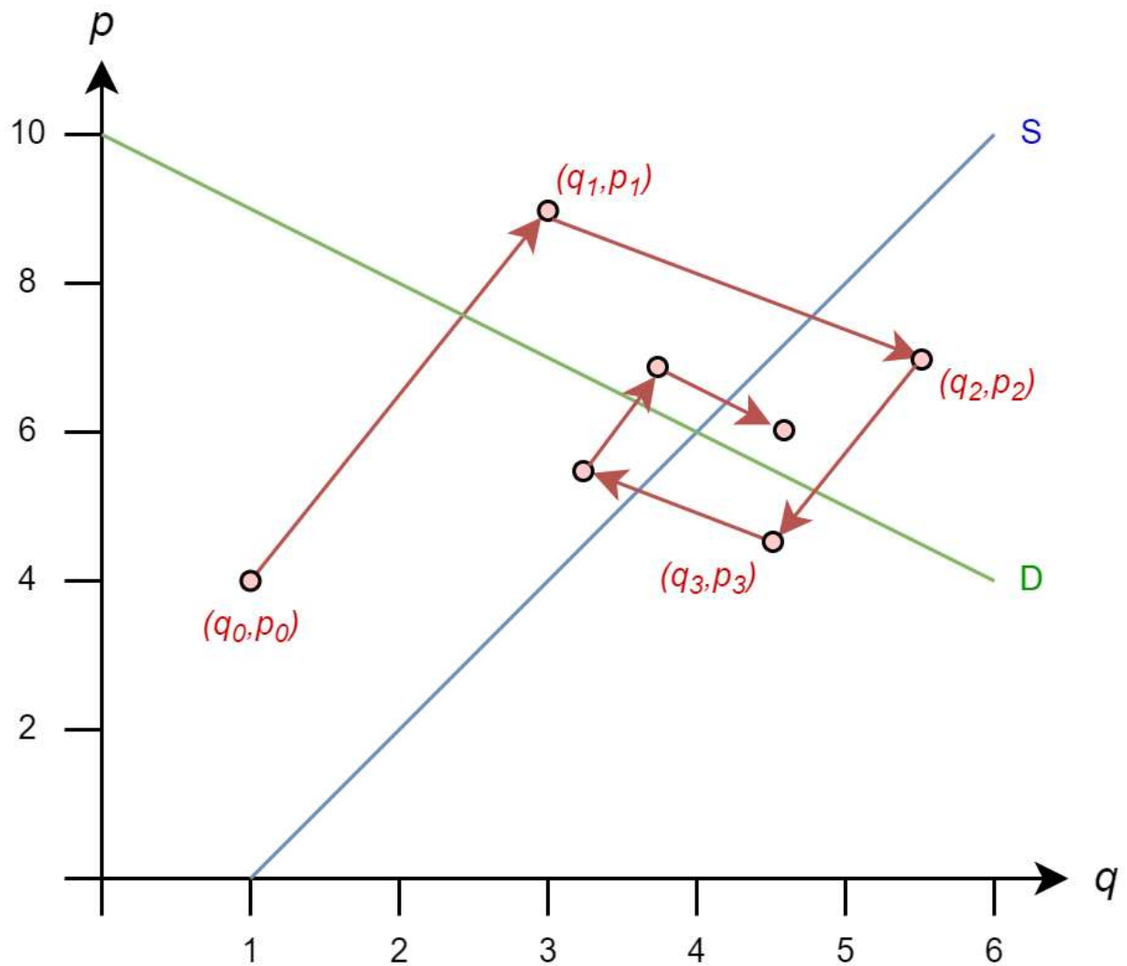
A visual example: Gauss-Jacobi

The iteration rules are

$$q^{(k+1)} = 1 + p^{(k)}/2$$

$$p^{(k+1)} = 10 - q^{(k)}$$

And we continue to process until the difference between $(q^{(k+1)}, p^{(k+1)})$ and $(q^{(k)}, p^{(k)})$ is smaller than our `tolerance` parameter (i.e., it converges)



A visual example: Gauss-Seidel

The Gauss-Seidel iteration rules looks similar, but there is an important difference

$$q^{(k+1)} = 1 + p^{(k)}/2$$
$$p^{(k+1)} = 10 - \mathbf{q^{(k+1)}}$$

$p^{(k+1)}$ is a function of $q^{(k+1)}$, not $q^{(k)}$

- Note that we could plug the 1st equation into the 2nd: $p^{(k+1)} = 9 - p^{(k)}/2$
- In this case, the example starts with an iteration over q. Starting with p is also possible *but gives a different sequence of steps*

A visual example: Gauss-Seidel

Let's see how that rule comes from matrix form

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \Rightarrow Q = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \Rightarrow Q^{-1} = \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix}$$

So $x^{(k+1)} = Q^{-1}b + (I - Q^{-1}A)x^{(k)}$ gives

$$\begin{bmatrix} p^{(k+1)} \\ q^{(k+1)} \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 10 \\ -2 \end{bmatrix} + \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \right) \begin{bmatrix} p^{(k)} \\ q^{(k)} \end{bmatrix}$$

$$\begin{bmatrix} p^{(k+1)} \\ q^{(k+1)} \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/2 & 0 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} p^{(k)} \\ q^{(k)} \end{bmatrix}$$

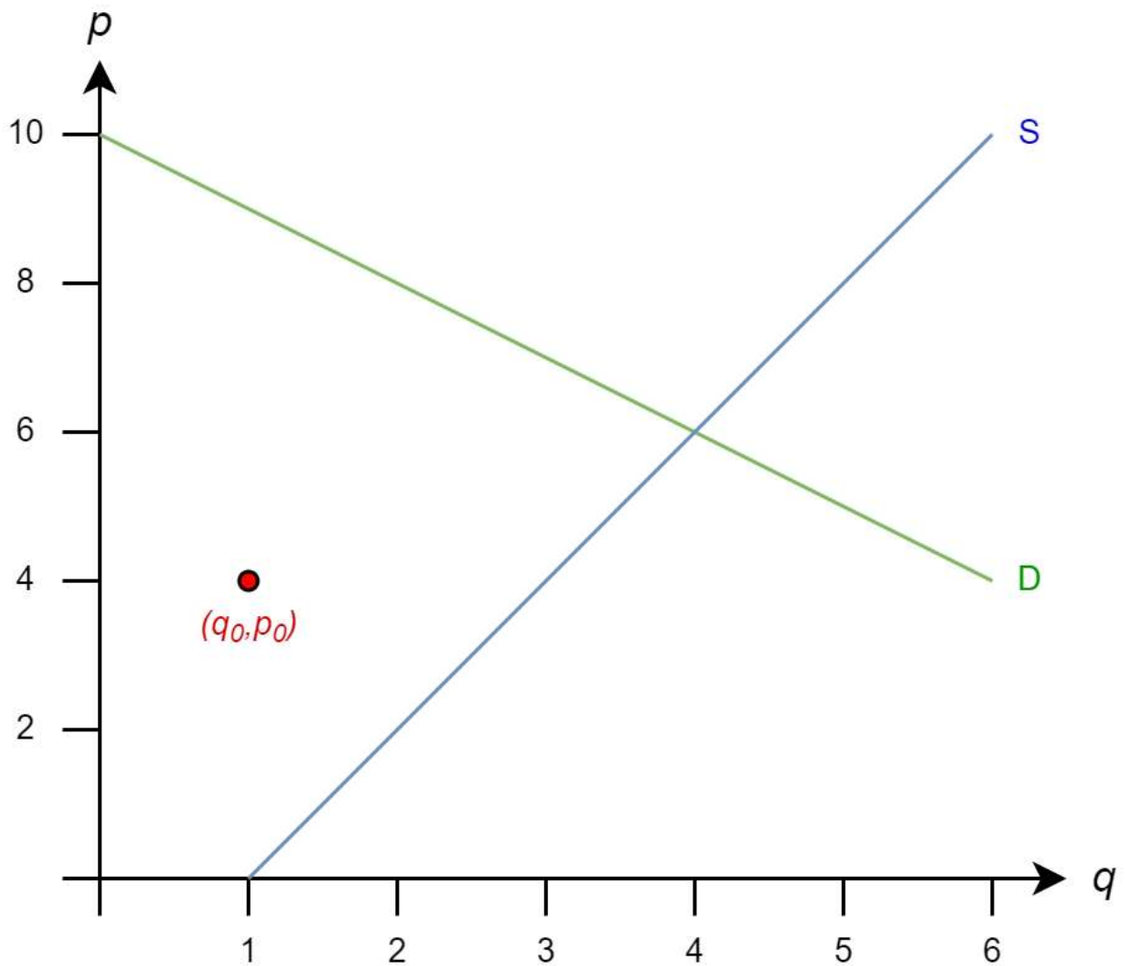
A visual example: Gauss-Seidel

Once again, we start from initial guess $q_0 = 1$ and $p_0 = 4$

The iteration rules are

$$q^{(k+1)} = 1 + p^{(k)}/2$$

$$p^{(k+1)} = 10 - q^{(k+1)}$$



A visual example: Gauss-Seidel

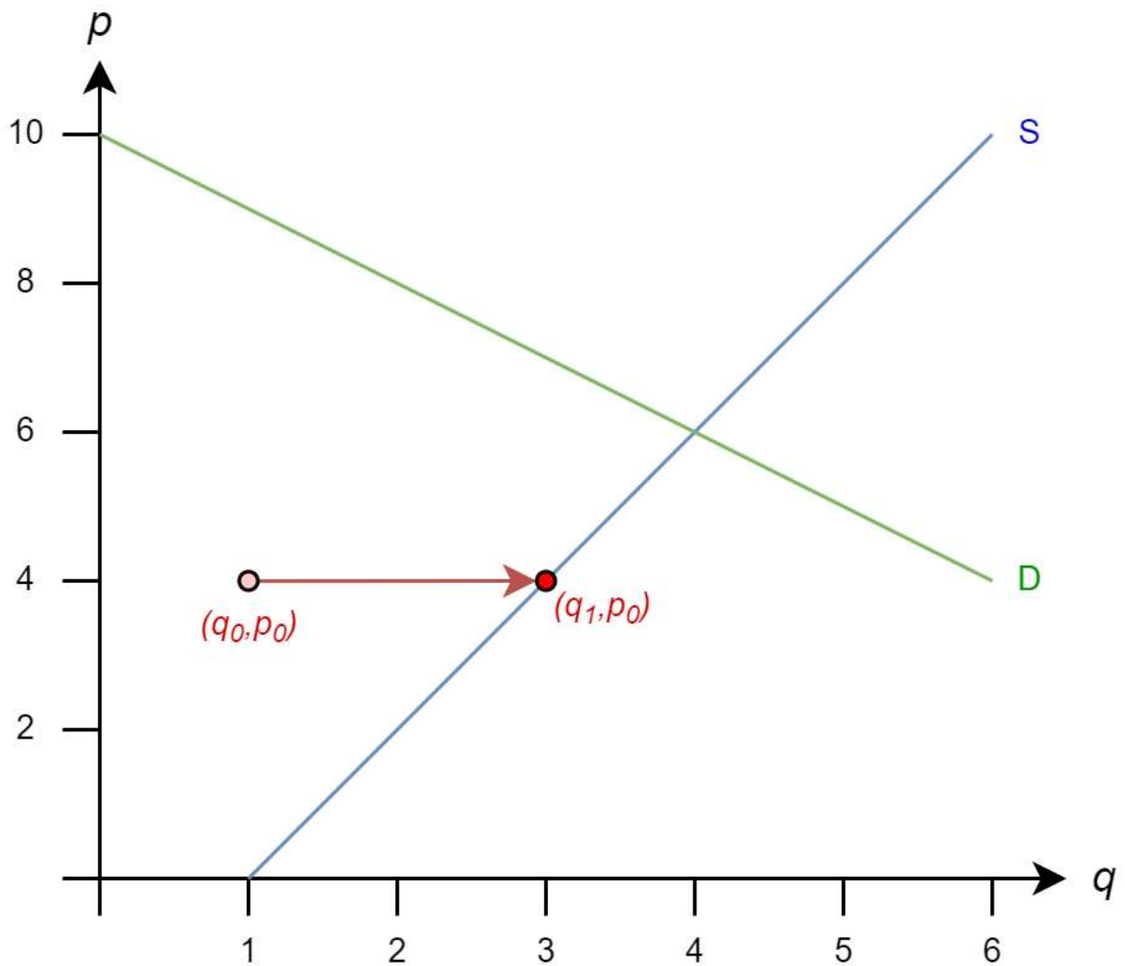
The iteration rules are

$$q^{(k+1)} = 1 + p^{(k)}/2$$

$$p^{(k+1)} = 10 - q^{(k+1)}$$

So

$$q_1 = 1 + (4)/2 = 3$$



A visual example: Gauss-Seidel

The iteration rules are

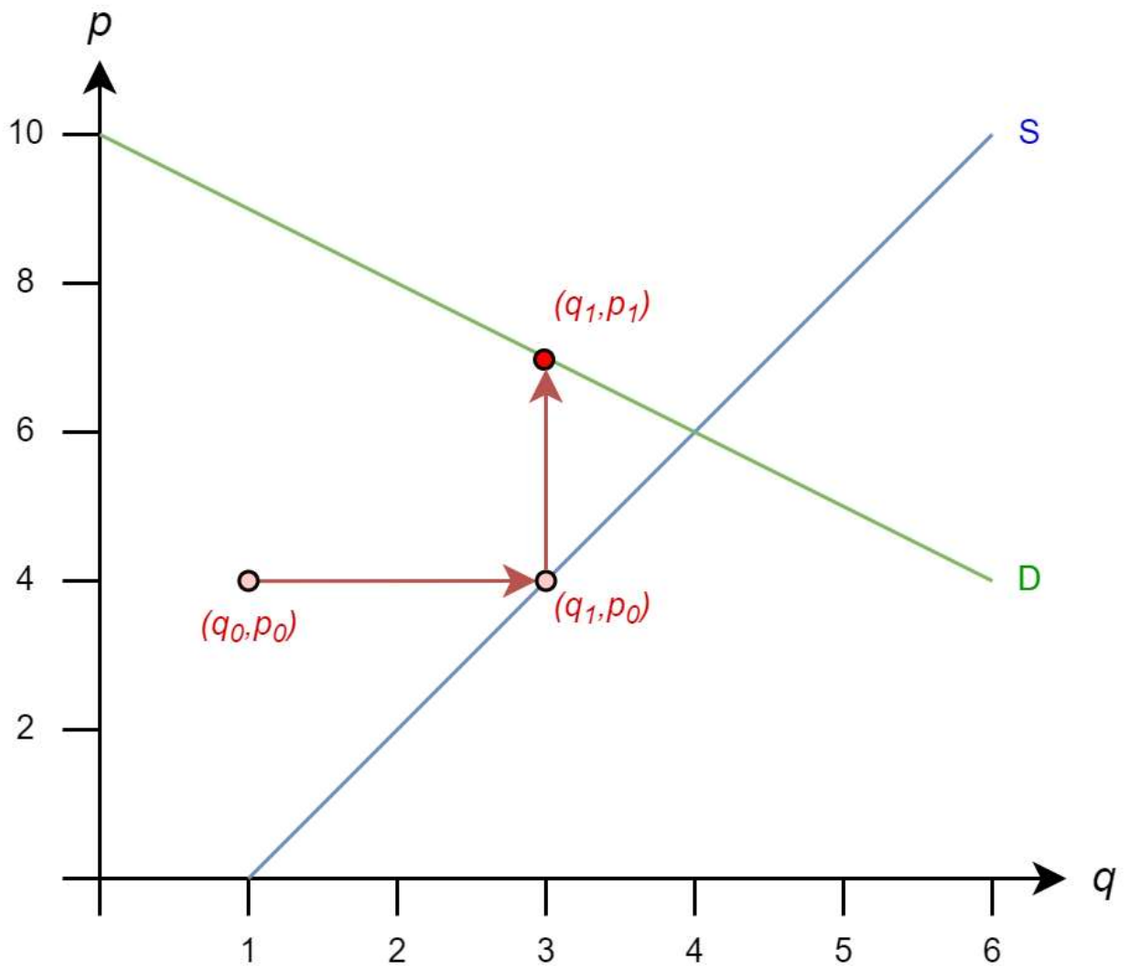
$$q^{(k+1)} = 1 + p^{(k)}/2$$

$$p^{(k+1)} = 10 - q^{(k+1)}$$

So

$$q_1 = 1 + (4)/2 = 3$$

$$p_1 = 10 - (3) = 7$$



A visual example: Gauss-Seidel

The iteration rules are

$$q^{(k+1)} = 1 + p^{(k)}/2$$

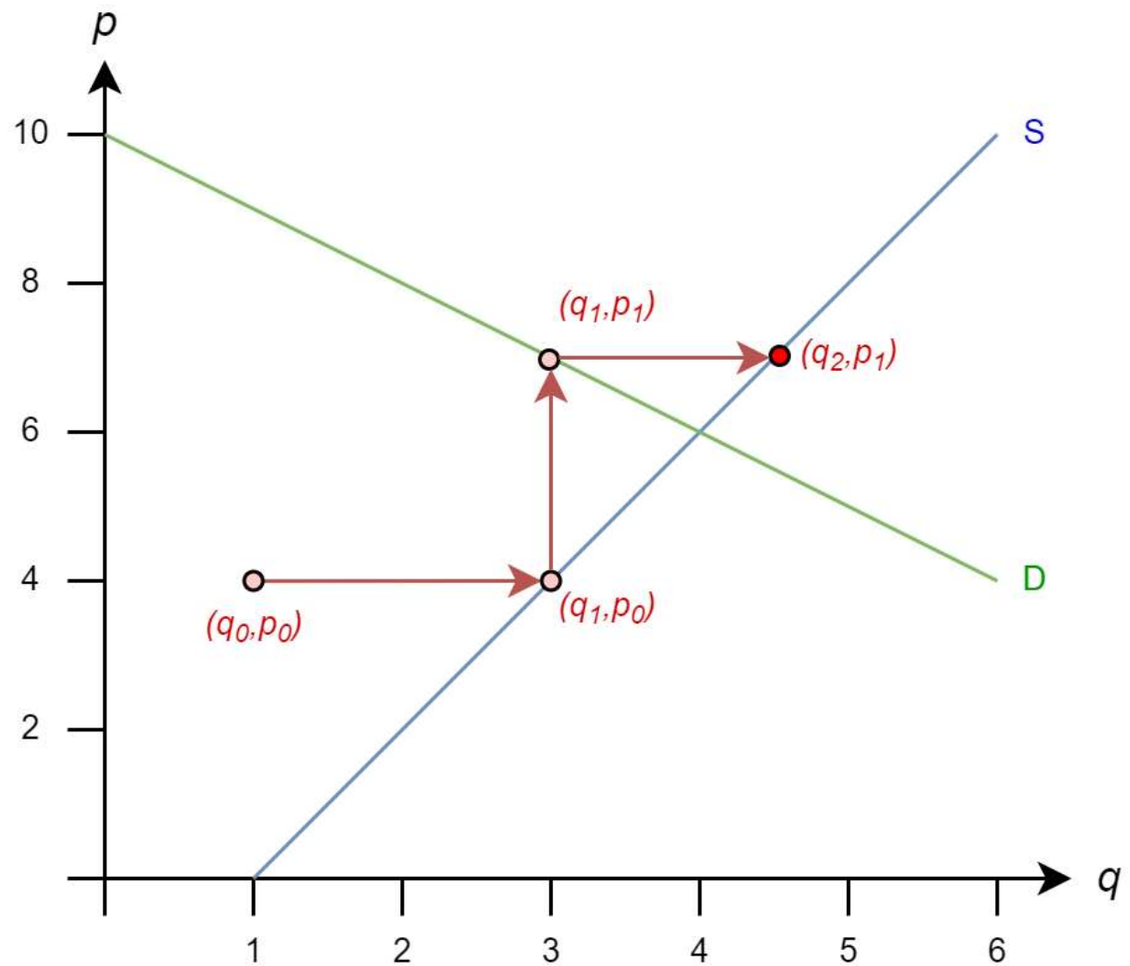
$$p^{(k+1)} = 10 - q^{(k+1)}$$

So

$$q_1 = 1 + (4)/2 = 3$$

$$p_1 = 10 - (3) = 7$$

$$q_2 = 1 + (7)/2 = 4.5$$



A visual example: Gauss-Seidel

The iteration rules are

$$q^{(k+1)} = 1 + p^{(k)}/2$$

$$p^{(k+1)} = 10 - q^{(k+1)}$$

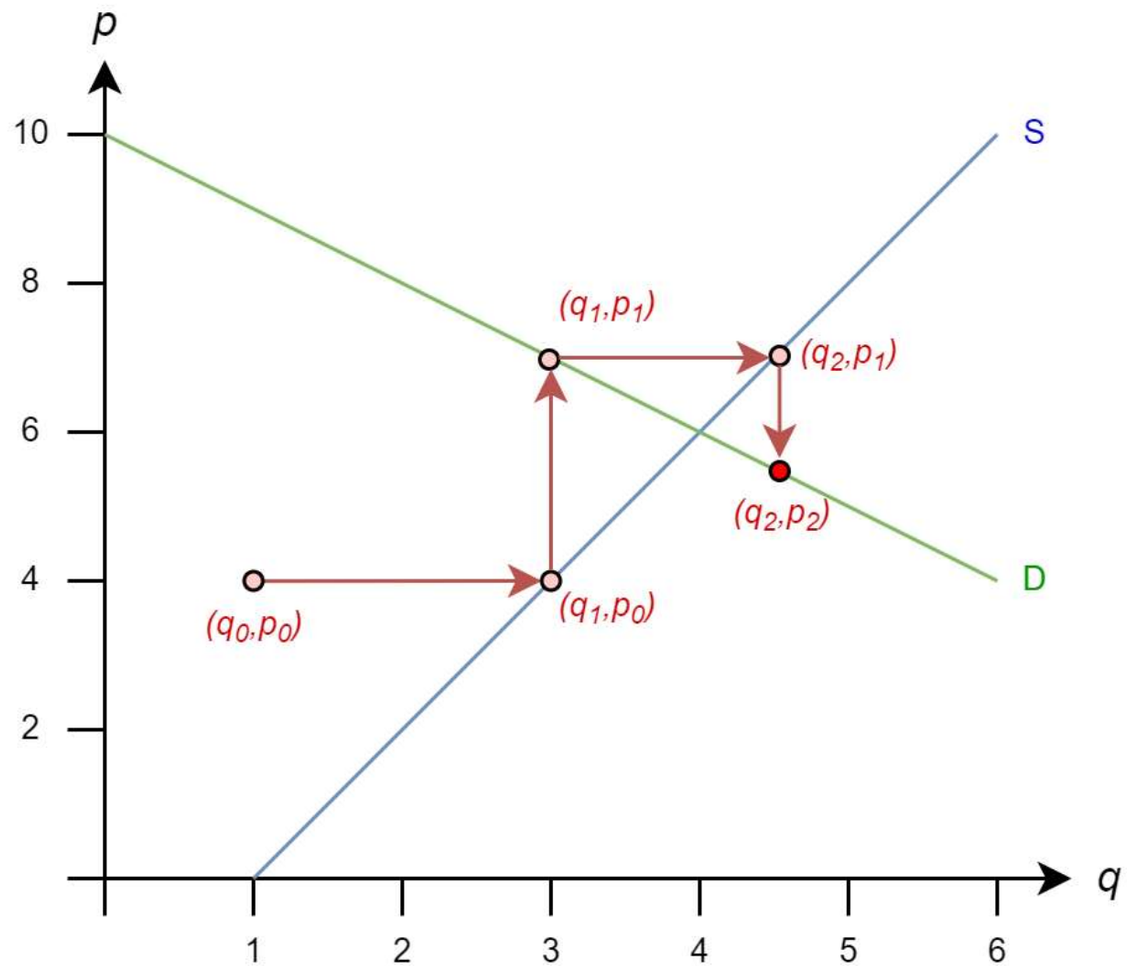
So

$$q_1 = 1 + (4)/2 = 3$$

$$p_1 = 10 - (3) = 7$$

$$q_2 = 1 + (7)/2 = 4.5$$

$$p_2 = 10 - 4.5 = 5.5$$



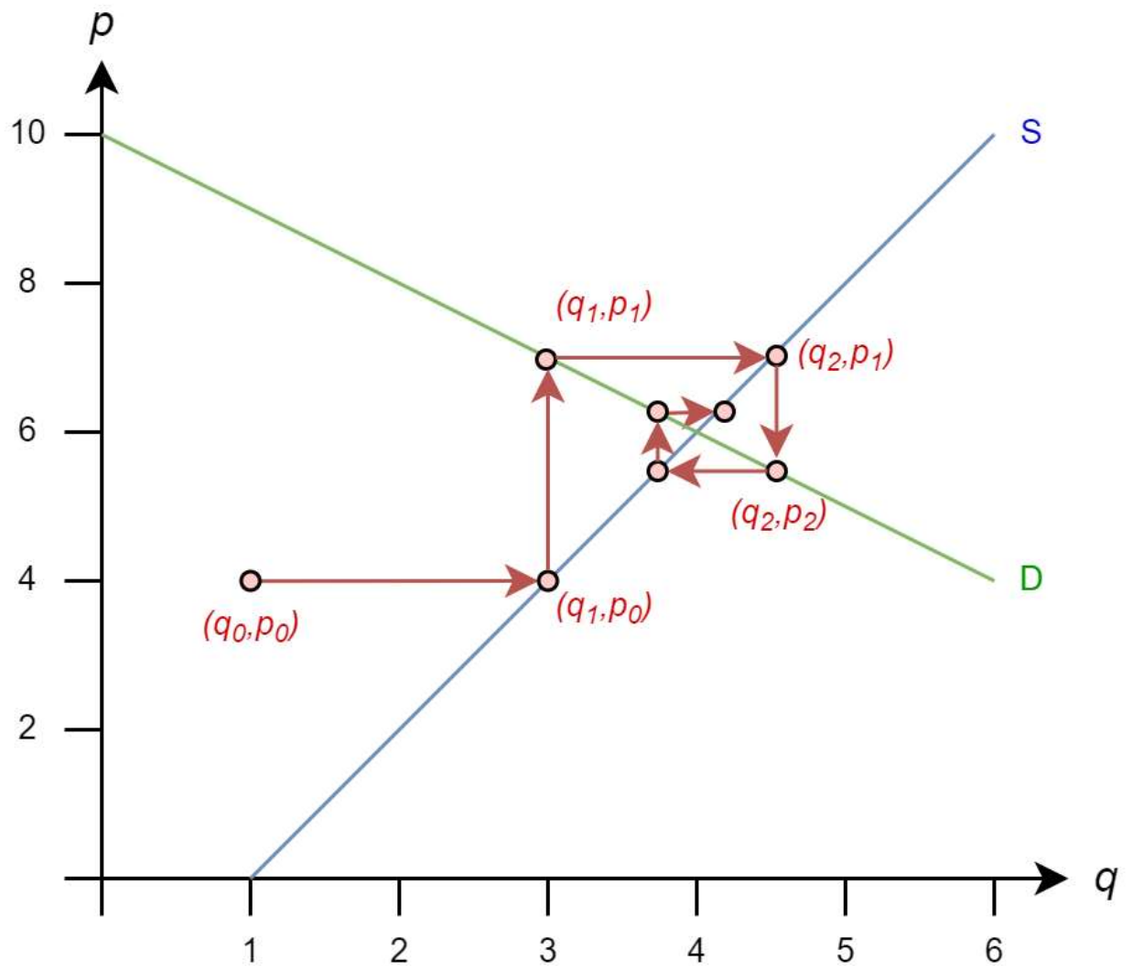
A visual example: Gauss-Seidel

The iteration rules are

$$q^{(k+1)} = 1 + p^{(k)}/2$$

$$p^{(k+1)} = 10 - q^{(k+1)}$$

And we continue to process until the difference between $(q^{(k+1)}, p^{(k+1)})$ and $(q^{(k)}, p^{(k)})$ is smaller than our `tolerance` parameter (i.e., it converges)



Acceleration and Stabilization methods

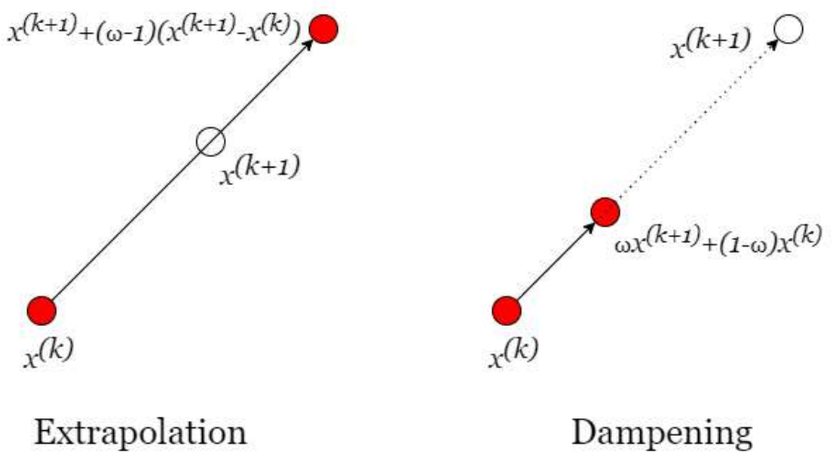
When our iterations converge too slowly or diverge, we can try **acceleration** or **stabilization** methods

- These concepts also work for nonlinear equations and optimization

These methods are conceptually simple: we will multiply our step $x^{(k)} \to x^{(k+1)}$ by a scalar ω

Acceleration and Stabilization methods

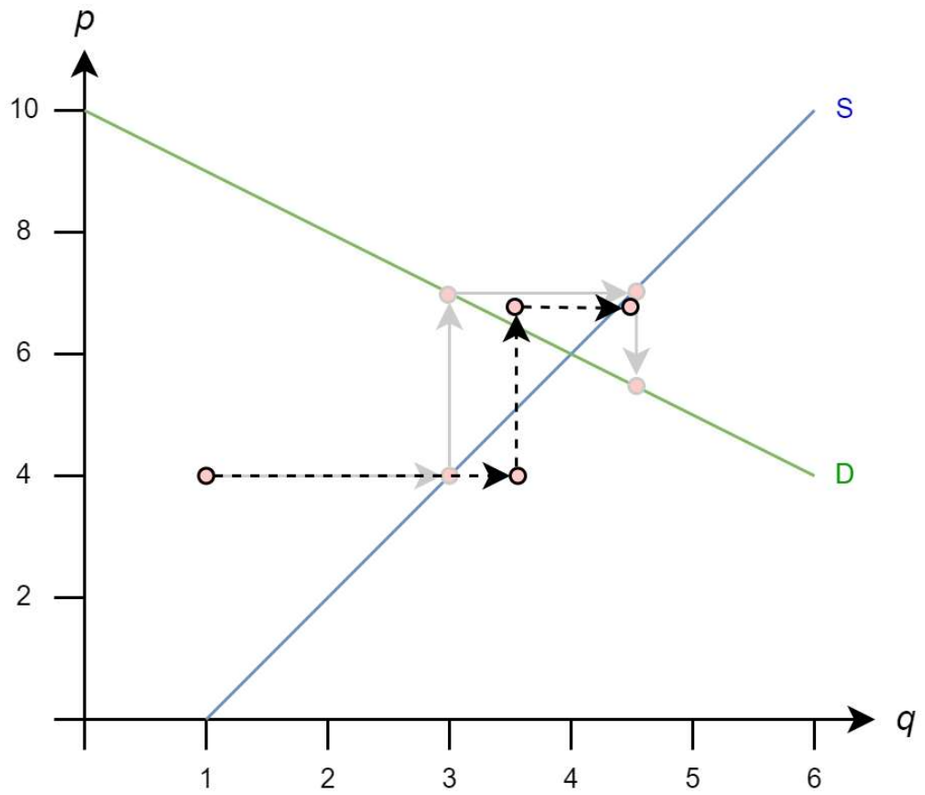
- When $\omega > 1$, it's called **extrapolation**: we're taking a *bigger step*
  - Idea: if the method converges, then the direction $x^{(k)} \to x^{(k+1)}$ is a good move, so it could be even better to go beyond $x^{(k+1)}$ and converge faster
- When $\omega < 1$, it's called **dampening**: we're taking a *smaller step*
  - Idea: maybe the direction $x^{(k)} \to x^{(k+1)}$ is a good one, but we are overshooting, so if we stop short of $x^{(k+1)}$, we may get it to converge



Extrapolation Dampening

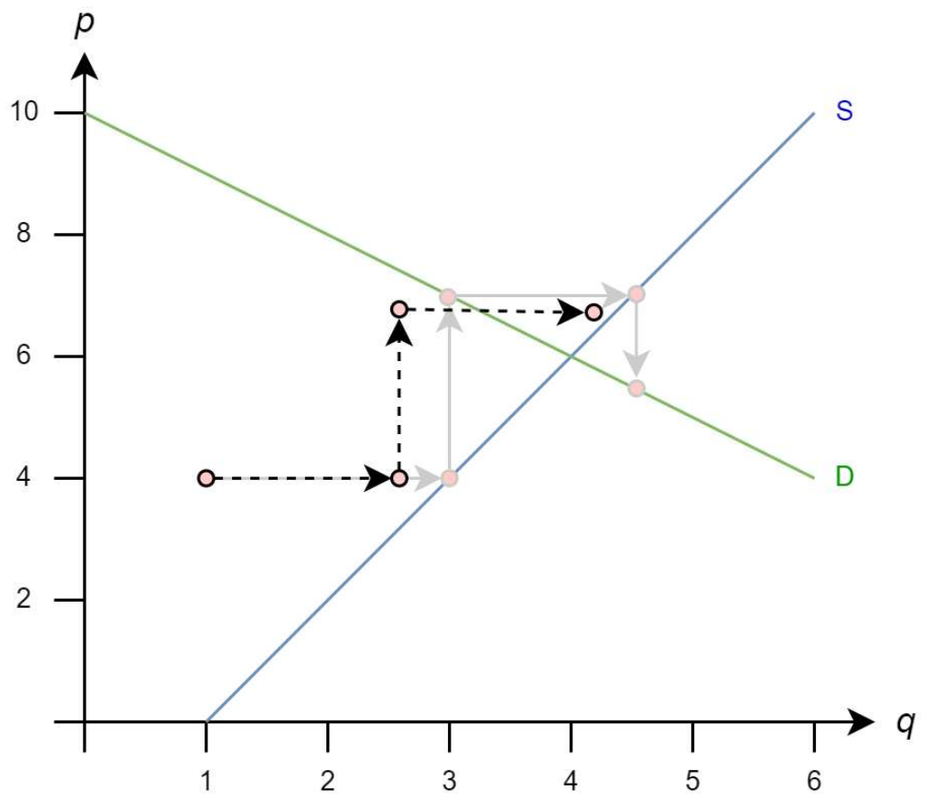
Acceleration and Stabilization methods

Returning to our Gauss-Seidel example, **accelerating** the solution with **extrapolation** could look like this



Acceleration and Stabilization methods

And, **stabilizing** the solution with **dampening** could look like this



Acceleration and Stabilization methods

These methods don't always work, may it's a good idea to try if you are having issues with slow convergence or divergence

We'll skip the technical details of when these methods work for operator splitting

- You can find them in chapter 3 of Judd's textbook

Course roadmap

Up next

1. Intro to Scientific Computing
2. Numerical operations and representations
3. **Systems of equations**
    1. Linear equations
    2. **Nonlinear equations** ←
4. Optimization
5. Structural estimation