Conjectures on the Period Lengths of One-Dimensional Cut-and-Project Sequences

Dirk Kunert *

April 28, 2025

Abstract

With $\alpha, \beta \in \mathbb{N}$, $\alpha \perp \beta$, $x, y \in \mathbb{R}$, $\omega \in \mathbb{R}_{\geq 0}$ and $i \in \mathbb{Z}$, we consider the points

$$\mathbf{C} = \left(\begin{pmatrix} x \\ y \end{pmatrix} \middle| \; x \in \mathbb{Z},\, y \in \left[\frac{\alpha}{\beta}(x - \omega),\; \frac{\alpha}{\beta}x + \omega \right] \cap \mathbb{Z} \right) \tag{1}$$

projected vertically onto $f(x) = \frac{\alpha}{\beta}x$ and measure their euclidian distances $\left(d^{(i)}\right)$. With

$$\Lambda_{\alpha,\beta} \; := \; \alpha + \beta \; + \; 1,$$

we propose the following conjectures concerning the period length λ of $\left(d^{(i)}\right)$:

1. There is always a finite λ.
2. If $\omega \in (0, 1)$, $\lambda_{\alpha,\beta} < \Lambda_{\alpha,\beta}$.
3. If $\omega = 1$, $\lambda_{\alpha,\beta} = \Lambda_{\alpha,\beta}$.
4. If $\omega \in (1, 2)$, $\lambda_{\alpha,\beta} \geq \Lambda_{\alpha,\beta}$.
5. If $\omega > 2$, $\lambda_{\alpha,\beta} > \Lambda_{\alpha,\beta}$.
6. If $\omega > 0$, $\lambda_{\alpha,\beta} \approx \lfloor \omega \, \Lambda_{\alpha,\beta} \rfloor$.

We use computational methods to support these conjectures.

Remark 1. We will show, that $\lambda_{\omega=0} = 1$.

Remark 2. Conjecture 6 is the result of a conversation with *OpenAI ChatGPT o3-mini-high* (see [4]).

1 Introduction

In 2017, Yves Meyer (École Normale Supérieure Paris-Saclay, France) won the Abel Prize “for his pivotal role in the development of the mathematical theory of wavelets” (see [1]). Terence Tao (University of California, Los Angeles) held the announcement (see [2]) and presented Figure 1. With regard to the distances between the projected points on the right side, he noted the following: ”They repeat themselves, but not in a regular fashion.”

These one-dimensional cut-and-project sequences are explored here. C and Python software is used to support our conjectures (see [6]).

2 Construction of the Sequences

2.1 Projection

In the figures 1 and 2, the functions that are projected vertically onto are $f(x) = \frac{\alpha}{\beta}x$, the upper, passing through $(0, \omega)$, are given by $u(x) = \frac{\alpha}{\beta}x + \omega$, and the lower, passing through $(\omega, 0)$, by $l(x) = \frac{\alpha}{\beta}(x - \omega)$, with $x \in \mathbb{R}$, $\alpha, \beta \in \mathbb{N}$, and the offsets $\omega \in \mathbb{R}_{\geq 0}$.

*Independent Researcher, Email: dirk.kunert@gmail.com

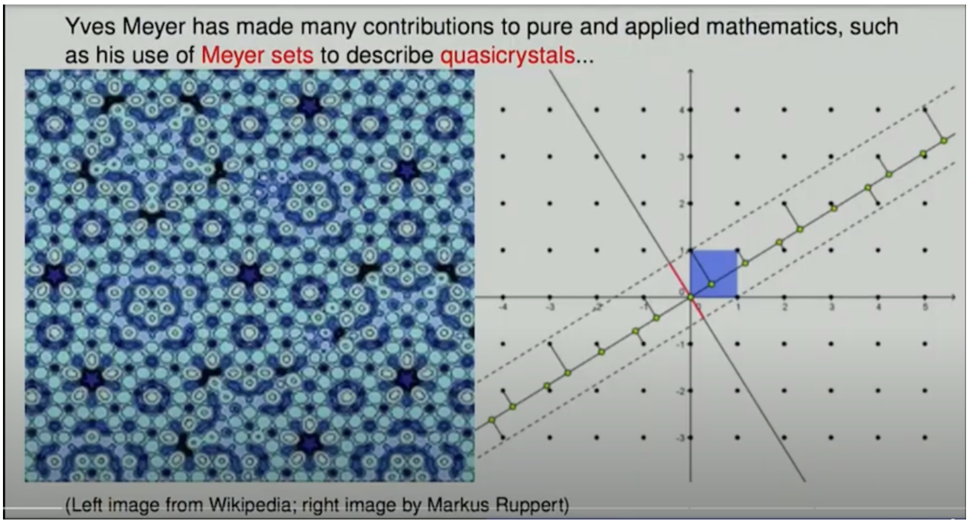

Figure 1: Screenshot from Terence Tao's presentation taken 60 seconds after the video's start (see [2])

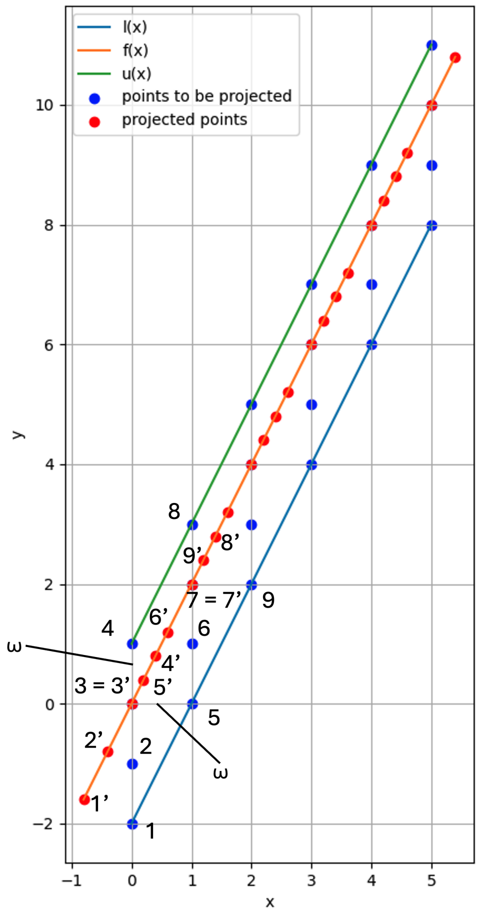

Figure 2: Illustration of the projection of points for $\frac{\alpha}{\beta} = 2$ and $\omega = 1$ within the interval $[0, 5]$

Projecting the points $\mathbf{C}$ described in equation 1 vertically on $f(x) = \frac{\alpha}{\beta}x$, we get

$$\mathbf{P} = \frac{1}{\alpha^2 + \beta^2} \begin{pmatrix} \beta^2 & \alpha\beta \\ \alpha\beta & \alpha^2 \end{pmatrix} \mathbf{C}. \tag{2}$$

Figure 2 shows that the first projected points are $1'$, $2'$, $3'$, $5'$, $4'$, $6'$, $7'$, $9'$, and $8'$. They are not in the correct order. So, $\mathbf{P}$ must be sorted by the x-coordinates of its points. We get

$$\mathbf{P}_s = \left(\mathbf{p}_s^{(i)}\right), \tag{3}$$

with $i \in \mathbb{Z}$.

Remark 3. We use $\mathbb{Z}$ instead of $\mathbb{N}$, because we have a "two-way infinite sequence" (see [3], p. 106).

Remark 4. Computational experiments show, that sorting has no influence on λ (see C function `test_sorting` in [6]).

For the analysis we limit x to $[0, x_{max}]$, with $x_{max} \in \mathbb{N}$. Examining Figure 2 again, we observe the following squared distances:

Index	<u>1</u>	2	3	4	5	...	18	19	20	21	22	<u>23</u>
Value	0.8	0.8	0.2	1.48	0.2	...	0.8	0.2	1.48	0.2	0.8	0.8

Table 1: Squared distances in Figure 2 for $x_{max} = 5$

By selecting the interval $[0, 5]$, 0.8 appears at index 1 and index 23, making the sequence appear non-periodic despite the repetition of $(0.8, 0.2, 1.48, 0.2)$. However, if we choose the interval $[0, 10]$, the sequence continues as expected after index 22:

Index	...	22	<u>23</u>	24	25	...
Value	...	0.8	0.2	1.48	0.2	...

Table 2: Squared distances in Figure 2 for $x_{max} = 10$

To solve this problem in the computational experiments, we delete elements from the beginning and the end of $\mathbf{P}_s$, while no finite λ is found. We limit the number of removed elements to 10% of the sequence length. This way we get K projected points.

2.2 Distance Calculation

The euclidian distances of the ordered projected points are

$$d^{(i)} = \left\| \mathbf{p}_s^{(i+1)} - \mathbf{p}_s^{(i)} \right\|. \tag{4}$$

Every injective function applied to the elements of $\left(d^{(i)}\right)_{i=1}^{K-1}$ does not change λ. $f(x) = x^2$ is injective for $x \in \mathbb{R}_{\geq 0}$. Thus, we can use the squared distances $\left(s^{(i)}\right)_{i=1}^{K-1}$ instead of the euclidian ones:

$$\begin{aligned} s^{(i)} &= \left(p_{xs}^{(i+1)} - p_{xs}^{(i)}\right)^2 + \left(p_{ys}^{(i+1)} - p_{ys}^{(i)}\right)^2 && (5) \\ &= \left(1 + \frac{\alpha^2}{\beta^2}\right) \left(p_{xs}^{(i+1)} - p_{xs}^{(i)}\right)^2, && (6) \end{aligned}$$

because

$$p_{ys}^{(i)} = \frac{\alpha}{\beta} p_{xs}^{(i)}. \tag{7}$$

$f(x) = \gamma x$ and $f(x) = \sqrt{x}$ are injective for $\gamma \in \mathbb{R}_{\neq 0}$ and $x \in \mathbb{R}_{\geq 0}$ as well. We get

$$\tilde{\delta}^{(i)} = p_{xs}^{(i+1)} - p_{xs}^{(i)}. \tag{8}$$

Equation (2) gives

$$p_x^{(i)} = \frac{\beta}{\alpha^2 + \beta^2} \left(\beta c_x^{(i)} + \alpha c_y^{(i)} \right), \tag{9}$$

where we can omit the constant $\frac{\beta}{\alpha^2+\beta^2}$:

$$\tilde{p}_x^{(i)} = \beta c_x^{(i)} + \alpha c_y^{(i)}. \tag{10}$$

When we sort $\left(\tilde{p}_x^{(i)}\right)$, the result is

$$\delta^{(i)} = \tilde{p}_{xs}^{(i+1)} - \tilde{p}_{xs}^{(i)}. \tag{11}$$

$\left(\delta^{(i)}\right)_{i=1}^{K-1}$ is now examined for periodicity.

Remark 5. $\mathbf{p}_s^{(i)} = \mathbf{p}_s^{(j)}$ for $i, j \in \mathbb{Z}$ and $i \neq j$ may occur under specific conditions: For $\frac{\alpha}{\beta} = 1$ and $\omega = 1$ both points, $(0, 1)$ and $(1, 0)$, are projected to $(0.5, 0.5)$. Therefore $\left(\delta^{(i)}\right)_{i=1}^{K-1}$ is not discrete.

3 Periodicity

We consider the finite sequence $\left(e^{(i)}\right)_{i=1}^{L}$ with $L \in \mathbb{N}_{>1}$ to be periodic, if there exists a $\tilde{\lambda} \in [1, L//2]$ with

$$e^{(i+\tilde{\lambda})} = e^{(i)} \ \forall i \in [1, L - (L \bmod \tilde{\lambda})]. \tag{12}$$

The period length λ is defined as

$$\lambda := \min\left(\tilde{\lambda}^{(i)}\right). \tag{13}$$

Proof. $\lambda_{\omega=0} = 1$: As sorting is not necessary, we have

$$\mathbf{p}_s^{(i)} = \mathbf{p}^{(i)} = i \begin{pmatrix} \beta \\ \alpha \end{pmatrix}. \tag{14}$$

If we use equation (8), we obtain the constants

$$\delta^{(i)} = \beta. \tag{15}$$

Hence $\lambda_{\omega=0} = 1$. □

4 Conjectures

In very rare cases, we see that our C software is not able to calculate a finite period length because the size of the array containing the differences is not sufficient. The array used has 8,000,000,000 64 bit elements and a size of approximately 60 GB.

4.1 Conjectures 1 to 5

Using the C software, we were able to support all conjectures listed in the abstract.

4.2 Conjecture 6

We found conjecture 6 by providing *OpenAI ChatGPT o3-mini-high* data for $\omega > 0$ created by the C function `create_test_data` and telling it, it should use *XGBoost* to predict λ. It created the Python function `xgboost` (see [6]).

When we sent the result of this computation, *OpenAI ChatGPT o3-mini-high* wrote of an "indication that the four predictors o_n, o_d, a_n, a_d [$\omega = \frac{o_n}{o_d}$, $\alpha = a_n$ and $\beta = a_d$] are in a highly complex, non-linear relationship, which *XGBoost* captures excellently". Then we reported on conjecture 3 and told *OpenAI ChatGPT o3-mini-high*, that we want to find a formula for λ.

Among other suggestions, it proposed the following:

$$\lambda = \omega\ \Lambda_{\alpha,\beta} \tag{16}$$

By experimenting with the C software we found that

$$\lambda = \lfloor \omega\ \Lambda_{\alpha,\beta} \rfloor \tag{17}$$

gives even better results. The lowest value we found for R^2 with the C function `test_conjectures` was 0.987994.

Acknowledgements

The author gratefully acknowledges the use of *OpenAI ChatGPT* (see [4]) for help refining the English phrasing and for suggestions on software implementation details. Except for conjecture 6, all conceptual content, data analysis, and the final manuscript were authored and verified by the present author.

In addition, the *DeepL Translator* (see [5]) has been used to improve the writing style.

References

[1] *Page for Yves Meyer at the Abel Prize Homepage*, available at: `https://abelprize.no/abel-prize-laureates/2017`, accessed on December 15, 2024.

[2] *Terence Tao on Yves Meyer's work on Wavelets*, available at: `https://youtu.be/AnkinNVPjyw`, accessed on December 15, 2024.

[3] Marjorie Senechal, *Quasicrystals and geometry*, 2009. ISBN: 978-0-521-57541-6.

[4] OpenAI, *ChatGPT*, available at `https://chat.openai.com`, accessed regularly between October 2024 and April 2025.

[5] DeepL GmbH, *DeepL Translator*, available at `https://www.deepl.com`, accessed regularly between October 2024 and April 2025.

[6] Dirk Kunert, 2025, *Repository* cut-and-project, available at: `https://github.com/dkunert/cut-and-project`.