Conjectures on the Period Lengths of One-Dimensional Cut-and-Project Sequences

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April 25, 2025

Abstract

With $\alpha, \beta \in \mathbb{N}$, $\alpha \perp \beta$, $x, y \in \mathbb{R}$, $\omega \in \mathbb{R}_{\geq 0}$ and $i \in \mathbb{Z}$, we consider the points

$$\mathbf{P} = \left(\begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, x \in \mathbb{Z}, \, y \in \left[\frac{\alpha}{\beta}(x - \omega), \, \frac{\alpha}{\beta}x + \omega \right] \cap \mathbb{Z} \right) \tag{1}$$

projected vertically onto $f(x) = \frac{\alpha}{\beta}x$ and measure their euclidian distances $\left(d^{(i)} \right)$. With

$$\Lambda_{\alpha,\beta} := \alpha + \beta + 1,$$

we propose the following conjectures concerning the period length λ of $\left(d^{(i)} \right)$:

1. There is always a finite λ.
2. If $\omega \in (0, 1)$, $\lambda_{\alpha,\beta} < \Lambda_{\alpha,\beta}$
3. If $\omega = 1$, $\lambda_{\alpha,\beta} = \Lambda_{\alpha,\beta}$.
4. If $\omega \in (1, 2)$, $\lambda_{\alpha,\beta} \geq \Lambda_{\alpha,\beta}$
5. If $\omega > 2$, $\lambda_{\alpha,\beta} > \Lambda_{\alpha,\beta}$
6. If $\omega \neq 1$, $\lambda \approx \lfloor (\omega \, \Lambda_{\alpha,\beta} \rfloor$

We use computational methods to support these conjectures.

Remark 1. We will show, that $\lambda_{\omega=0} = 1$.

Remark 2. Conjecture 6 is the result of a conversation with *OpenAI ChatGPT o3-mini-high* (see [4]).

1 Introduction

In 2017, Yves Meyer (École Normale Supérieure Paris-Saclay, France) won the Abel Prize "for his pivotal role in the development of the mathematical theory of wavelets" (see [1]). Terence Tao (University of California, Los Angeles) held the announcement (see [2]) and presented Figure 1.

Observing the distances of the points (see Figure 1, right), he stated the following: "They repeat themselves, but not in a regular fashion."

These one-dimensional cut-and-project sequences are explored here. C and Python software is used to support our conjectures (see [6]).

2 Construction of the Set

2.1 Projection

In Figure 2, the function that is projected vertically onto is $f(x) = \frac{\alpha}{\beta}x$, the upper, passing through $(0, \omega)$, is given by $u(x) = \frac{\alpha}{\beta}x + \omega$, and the lower, passing through $(\omega, 0)$, by $l(x) = \frac{\alpha}{\beta}(x - \omega)$, with $x \in \mathbb{R}$, $\alpha, \beta \in \mathbb{N}$, and the offset $\omega \in \mathbb{R}_{\geq 0}$.

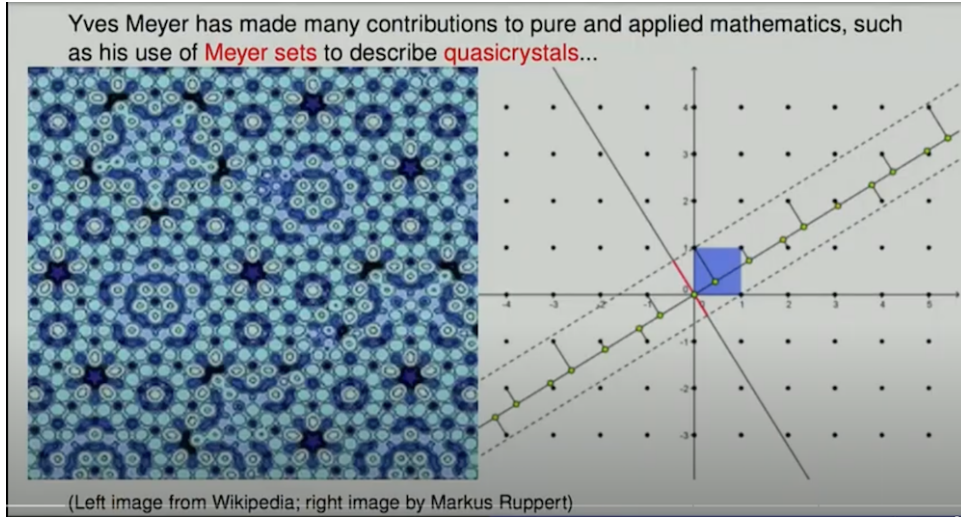[*]Independent Researcher, Email: dirk.kunert@gmail.com

Figure 1: Screenshot from Terence Tao's presentation taken 60 seconds after the video's start
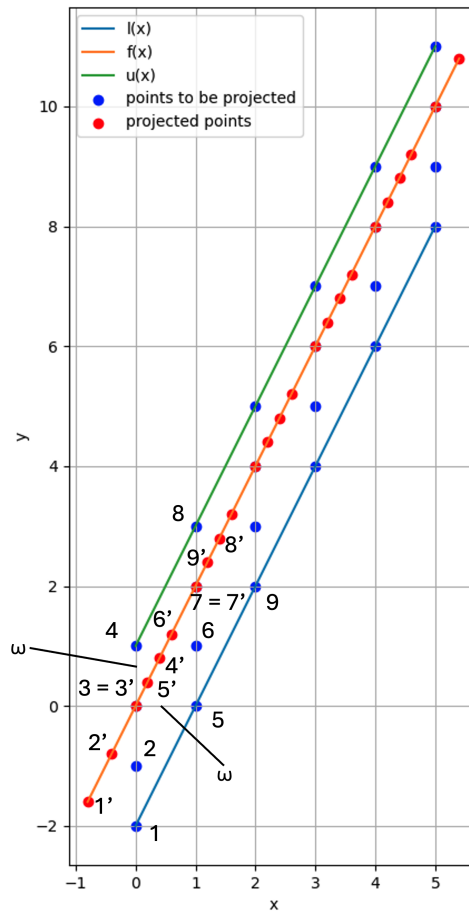

Figure 2: Illustration of the projection of points for $\frac{\alpha}{\beta} = 2$ and $\omega = 1$ within the interval $[0, 5]$

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Projecting the points shown in equation 1 vertically on $f(x) = \frac{\alpha}{\beta}x$, we get

$$\tilde{\mathbf{P}} = \frac{1}{\alpha^2 + \beta^2} \begin{pmatrix} \beta^2 & \alpha\beta \\ \alpha\beta & \alpha^2 \end{pmatrix} \mathbf{P}. \tag{2}$$

Figure 2 shows that the first projected points are $1'$, $2'$, $3'$, $5'$, $4'$, $6'$, $7'$, $9'$, and $8'$. They are not in the correct order. So, $\tilde{\mathbf{P}}$ must be sorted by the x-coordinates of its points. We get

$$\tilde{\mathbf{P}}_s = \left(\tilde{\mathbf{p}}_s^{(i)} \right), \tag{3}$$

with $i \in \mathbb{Z}$.

Remark 3. We use $\mathbb{Z}$ instead of $\mathbb{N}$, because we have a "two-way infinite sequence" (see [3], p. 106).

Remark 4. Computational experiments show, that sorting has no influence on λ.

For the analysis we limit x to $[0, x_{max}]$, with $x_{max} \in \mathbb{N}$. Examining Figure 2 again, we observe the following squared distances:

Index	1	2	3	4	5	...	18	19	20	21	22	23
Value	0.8	0.8	0.2	1.48	0.2	...	0.8	0.2	1.48	0.2	0.8	0.8

Table 1: Squared distances in Figure 2 for $x_{max} = 5$

By selecting the interval $[0, 5]$, 0.8 is appended at both the beginning and the end, making the sequence appear non-periodic despite the repetition of $(0.8, 0.2, 1.48, 0.2)$. However, if we choose the interval $[0, 10]$, the sequence continues as expected:

Index	...	22	23	24	25	...
Value	...	0.8	0.2	1.48	0.2	...

Table 2: Squared distances in Figure 2 for $x_{max} = 10$

To solve this problem in the computational experiments, we delete elements from the beginning and the end of P_s, while no finite λ is found. We limit the number of removed elements to 10% of the sequence length. This way we get K projected points.

2.2 Distance Calculation

The euclidian distances of the ordered projected points with $\tilde{p}_{xs}^{(i)} \in [0, x_{max}]$ are

$$\left(d^{(i)} \right)_{i=1}^{K-1} = \left\| \tilde{\mathbf{p}}_s^{(i+1)} - \tilde{\mathbf{p}}_s^{(i)} \right\|. \tag{4}$$

Every injective function applied to the elements of $\left(d^{(i)} \right)_{i=1}^{K-1}$ does not change λ. $f(x) = x^2$ is injective for $x \in \mathbb{R}_{\geq 0}$. Thus, we can use the squared distances $\left(s^{(i)} \right)_{i=1}^{K-1}$ instead of the euclidian ones:

$$s^{(i)} = \left(\tilde{p}_{xs}^{(i+1)} - \tilde{p}_{xs}^{(i)} \right)^2 + \left(\tilde{p}_{ys}^{(i+1)} - \tilde{p}_{ys}^{(i)} \right)^2 \tag{5}$$

$$= \left(1 + \frac{\alpha^2}{\beta^2} \right) \left(\tilde{p}_{xs}^{(i+1)} - \tilde{p}_{xs}^{(i)} \right)^2, \tag{6}$$

because

$$\tilde{p}_{ys}^{(i)} = \frac{\alpha}{\beta} \tilde{p}_{xs}^{(i)}. \tag{7}$$

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$f(x) = \gamma x$ and $f(x) = \sqrt{x}$ are injective for $\gamma \in \mathbb{R}_{\neq 0}$ and $x \in \mathbb{R}_{\geq 0}$ as well. We get

$$\tilde{s}^{(i)} = \left(\tilde{p}_{xs}^{(i+1)} - \tilde{p}_{xs}^{(i)} \right). \tag{8}$$

From equation (2) it follows that

$$\tilde{p}_{xs}^{(i)} = \frac{\beta}{\alpha^2 + \beta^2} \left(\beta p_{xs}^{(i)} + \alpha p_{ys}^{(i)} \right). \tag{9}$$

Here we also can omit the constant $\frac{\beta}{\alpha^2+\beta^2}$. Finally, we obtain

$$\left(\delta^{(i)} \right)_{i=1}^{K-1} = \left(\beta \left(p_{xs}^{(i+1)} - p_{xs}^{(i)} \right) + \alpha \left(p_{ys}^{(i+1)} - p_{ys}^{(i)} \right) \right)_{i=1}^{K-1}. \tag{10}$$

This sequence is now examined for periodicity.

Remark 5. $\tilde{p}_{xs}^{(i)} = \tilde{p}_{xs}^{(j)}$ for $i, j \in \mathbb{Z}$ and $i \neq j$ may occur under specific conditions: For $\frac{\alpha}{\beta} = 1$ and $\omega = 1$ both points, $(0, 1)$ and $(1, 0)$, are projected to $(0.5, 0.5)$. Therefore $\left(\delta^{(i)} \right)_{i=1}^{K-1}$ is not discrete.

3 Periodicity

We consider the finite sequence $\left(e^{(i)} \right)_{i=1}^{L}$ with $L \in \mathbb{N}_{>1}$ to be periodic, if there exists a $\tilde{\lambda} \in [1, L//2]$ with

$$e^{(i+\tilde{\lambda})} = e^{(i)} \ \forall i \in [1, L - (L \bmod \tilde{\lambda})]. \tag{11}$$

The period length λ is defined as

$$\lambda := \min \left(\tilde{\lambda}^{(i)} \right). \tag{12}$$

Proof. $\lambda_{\omega=0} = 1$: As sorting is not necessary, $\mathbf{P}_s = \mathbf{P}$. Thus,

$$\left(\delta^{(i)} \right)_{i=1}^{K-1} = \left(\beta \left(p_x^{(i+1)} - p_x^{(i)} \right) + \alpha \left(p_y^{(i+1)} - p_y^{(i)} \right) \right)_{i=1}^{K-1} \tag{13}$$

according to 10. As $p_x^{(i+1)} - p_x^{(i)} = 1$ and $p_y^{(i+1)} - p_y^{(i)} = \frac{\alpha}{\beta}$, we get

$$\left(\delta^{(i)} \right)_{i=1}^{K-1} = \left(\beta + \frac{\alpha^2}{\beta} \right)_{i=1}^{K-1}, \tag{14}$$

which consists of constant elements. As $e^{(i+1)} = e^{(i)} \ \forall i \in [1, K-2]$, $\lambda_{\omega=0} = 1$. $\qquad \square$

4 Conjectures

In very rare cases, we see that our C software is not able to calculate a finite period length because the size of the array containing the differences is not sufficient. The array used has 8,000,000,000 64 bit elements and a size of approximately 60 GB.

4.1 Conjectures 1 to 5

Using the C software, we were able to support all conjectures listed in the abstract.

4.2 Conjecture 6

We found conjecture 6 by providing *OpenAI ChatGPT o3-mini-high* data created by the C function `create_test_data` and telling it, it should use *XGBoost* to predict λ. It created the Python function `xgboost` (see [6]).

When we sent *OpenAI ChatGPT o3-mini-high* the result of this computation, it wrote of an "indication that the four predictors o_n, o_d, a_n, a_d [$\omega = \frac{o_n}{o_d}$, $\alpha = a_n$ and $\beta = a_d$] are in a highly complex, non-linear relationship, which *XGBoost* captures excellently". Then we reported on conjecture 3 and told *OpenAI ChatGPT o3-mini-high*, that we want to find a formula for λ.

Among other suggestions, it proposed the following:

$$\lambda = \omega \, \Lambda_{\alpha,\beta} \tag{15}$$

By experimenting with the C software we found that

$$\lambda = \lfloor \omega \, \Lambda_{\alpha,\beta} \rfloor \tag{16}$$

gives even better results. The lowest value we found for R^2 with the C function `test_conjectures` was 0.987994.

Acknowledgements

The author gratefully acknowledges the use of *OpenAI ChatGPT* (see [4]) for help refining the English phrasing and for suggestions on software implementation details. Except for conjecture 6, all conceptual content, data analysis, and the final manuscript were authored and verified by the present author.

In addition, *DeepL Translator* (see [5]) has been used to improve the writing style.

References

[1] *Page for Yves Meyer at the Abel Prize Homepage*, available at: `https://abelprize.no/abel-prize-laureates/2017`, accessed on December 15, 2024.

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[6] Dirk Kunert, 2025, *This article and the source code*, available at: `https://github.com/dkunert/cut-and-project/tree/main/src`.