

# Conjectures on the Period Lengths of One-Dimensional Cut-and-Project Sequences

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## Abstract

With  $\alpha, \beta \in \mathbb{N}$ ,  $\alpha \perp \beta$ ,  $x, y \in \mathbb{R}$ ,  $\omega \in \mathbb{R}_{\geq 0}$  and  $i \in \mathbb{Z}$ , we consider the points

$$\mathbf{C} = \left( \begin{pmatrix} x \\ y \end{pmatrix} \mid x \in \mathbb{Z}, y \in \left[ \frac{\alpha}{\beta}(x - \omega), \frac{\alpha}{\beta}x + \omega \right] \cap \mathbb{Z} \right) \quad (1)$$

projected vertically onto  $f(x) = \frac{\alpha}{\beta}x$  and measure their euclidian distances  $(d^{(i)})$ . With

$$\Lambda_{\alpha, \beta} := \alpha + \beta + 1,$$

we propose the following conjectures concerning the period length  $\lambda$  of  $(d^{(i)})$ :

1. There is always a finite  $\lambda$ .
2. If  $\omega \in (0, 1)$ ,  $\lambda_{\alpha, \beta} < \Lambda_{\alpha, \beta}$ .
3. If  $\omega = 1$ ,  $\lambda_{\alpha, \beta} = \Lambda_{\alpha, \beta}$ .
4. If  $\omega \in (1, 2)$ ,  $\lambda_{\alpha, \beta} \geq \Lambda_{\alpha, \beta}$ .
5. If  $\omega > 2$ ,  $\lambda_{\alpha, \beta} > \Lambda_{\alpha, \beta}$ .
6. If  $\omega > 0$ ,  $\lambda_{\alpha, \beta} \approx \lfloor \omega \Lambda_{\alpha, \beta} \rfloor$ .

We use computational methods to support these conjectures.

*Remark 1.* We will show, that  $\lambda_{\omega=0} = 1$ .

*Remark 2.* Conjecture 6 is the result of a conversation with *OpenAI ChatGPT o3-mini-high* (see [4]).

## 1 Introduction

In 2017, Yves Meyer (École Normale Supérieure Paris-Saclay, France) won the Abel Prize “for his pivotal role in the development of the mathematical theory of wavelets” (see [1]). Terence Tao (University of California, Los Angeles) held the announcement (see [2]) and presented Figure 1. With regard to the distances between the projected points on the right side, he noted the following: “They repeat themselves, but not in a regular fashion.”

These one-dimensional cut-and-project sequences are explored here. C and Python software is used to support our conjectures (see [6]).

## 2 Construction of the Sequences

### 2.1 Projection

In the figures 1 and 2, the functions that are projected vertically onto are  $f(x) = \frac{\alpha}{\beta}x$ , the upper, passing through  $(0, \omega)$ , are given by  $u(x) = \frac{\alpha}{\beta}x + \omega$ , and the lower, passing through  $(\omega, 0)$ , by  $l(x) = \frac{\alpha}{\beta}(x - \omega)$ , with  $x \in \mathbb{R}$ ,  $\alpha, \beta \in \mathbb{N}$ , and the offsets  $\omega \in \mathbb{R}_{\geq 0}$ .

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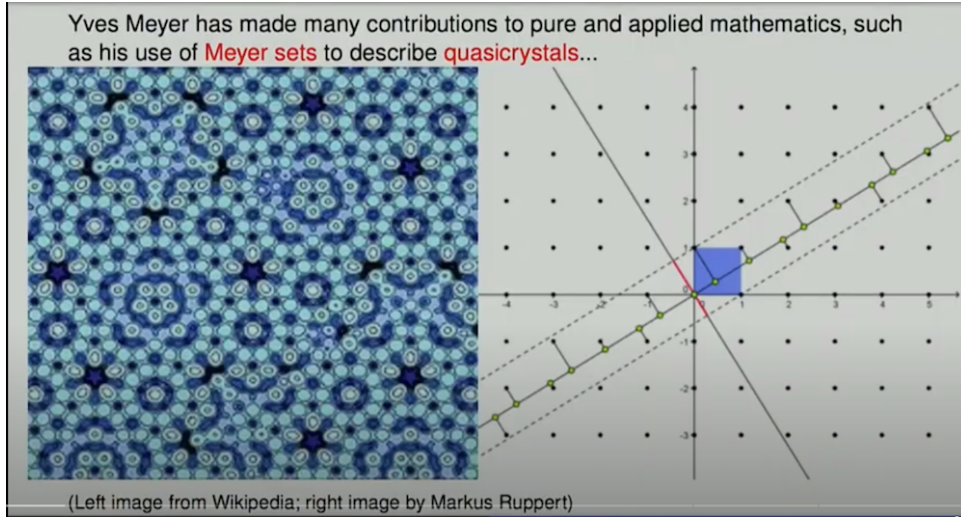


Figure 1: Screenshot from Terence Tao's presentation taken 60 seconds after the video's start (see [2])

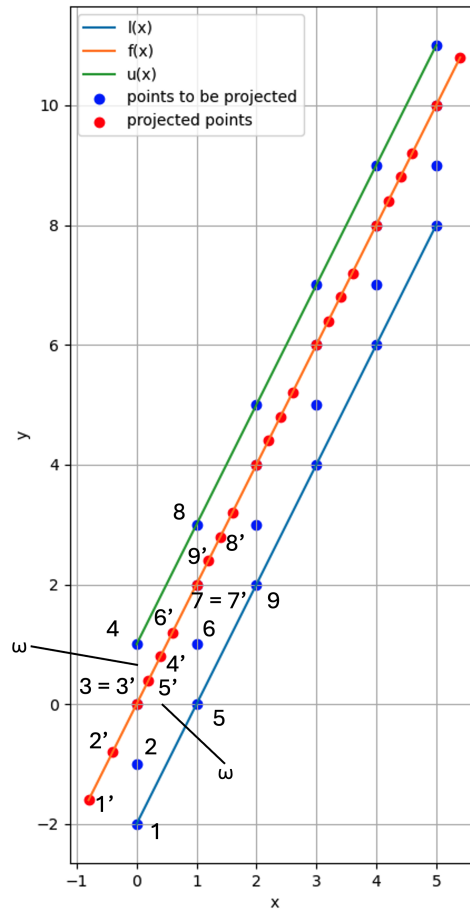


Figure 2: Illustration of the projection of points for  $\frac{\alpha}{\beta} = 2$  and  $\omega = 1$  within the interval  $[0, 5]$

Projecting the points  $\mathbf{C}$  described in equation 1 vertically on  $f(x) = \frac{\alpha}{\beta}x$ , we get

$$\mathbf{P} = \frac{1}{\alpha^2 + \beta^2} \begin{pmatrix} \beta^2 & \alpha\beta \\ \alpha\beta & \alpha^2 \end{pmatrix} \mathbf{C}. \quad (2)$$

Figure 2 shows that the first projected points are  $1', 2', 3', 5', 4', 6', 7', 9'$ , and  $8'$ . They are not in the correct order. So,  $\mathbf{P}$  must be sorted by the x-coordinates of its points. We get

$$\mathbf{P}_s = \left( \mathbf{p}_s^{(i)} \right), \quad (3)$$

with  $i \in \mathbb{Z}$ .

*Remark 3.* We use  $\mathbb{Z}$  instead of  $\mathbb{N}$ , because we have a "two-way infinite sequence" (see [3], p. 106).

*Remark 4.* Computational experiments show, that sorting has no influence on  $\lambda$ . **TODO: Check!**

For the analysis we limit  $x$  to  $[0, x_{max}]$ , with  $x_{max} \in \mathbb{N}$ . Examining Figure 2 again, we observe the following squared distances:

Index	<u>1</u>	2	3	4	5	...	18	19	20	21	22	<u>23</u>
Value	0.8	0.8	0.2	1.48	0.2	...	0.8	0.2	1.48	0.2	0.8	0.8

Table 1: Squared distances in Figure 2 for  $x_{max} = 5$

By selecting the interval  $[0, 5]$ , 0.8 appears at index 1 and index 23, making the sequence appear non-periodic despite the repetition of  $(0.8, 0.2, 1.48, 0.2)$ . However, if we choose the interval  $[0, 10]$ , the sequence continues as expected after index 22:

Index	...	22	<u>23</u>	24	25	...
Value	...	0.8	0.2	1.48	0.2	...

Table 2: Squared distances in Figure 2 for  $x_{max} = 10$

To solve this problem in the computational experiments, we delete elements from the beginning and the end of  $\mathbf{P}_s$ , while no finite  $\lambda$  is found. We limit the number of removed elements to 10% of the sequence length. This way we get  $K$  projected points.

## 2.2 Distance Calculation

The euclidian distances of the ordered projected points are

$$d^{(i)} = \left\| \mathbf{p}_s^{(i+1)} - \mathbf{p}_s^{(i)} \right\|. \quad (4)$$

Every injective function applied to the elements of  $(d^{(i)})_{i=1}^{K-1}$  does not change  $\lambda$ .  $f(x) = x^2$  is injective for  $x \in \mathbb{R}_{\geq 0}$ . Thus, we can use the squared distances  $(s^{(i)})_{i=1}^{K-1}$  instead of the euclidian ones:

$$s^{(i)} = \left( p_{xs}^{(i+1)} - p_{xs}^{(i)} \right)^2 + \left( p_{ys}^{(i+1)} - p_{ys}^{(i)} \right)^2 \quad (5)$$

$$= \left( 1 + \frac{\alpha^2}{\beta^2} \right) \left( p_{xs}^{(i+1)} - p_{xs}^{(i)} \right)^2, \quad (6)$$

because

$$p_{ys}^{(i)} = \frac{\alpha}{\beta} p_{xs}^{(i)}. \quad (7)$$

$f(x) = \gamma x$  and  $f(x) = \sqrt{x}$  are injective for  $\gamma \in \mathbb{R}_{\neq 0}$  and  $x \in \mathbb{R}_{\geq 0}$  as well. We get

$$\delta^{(i)} = p_{xs}^{(i+1)} - p_{xs}^{(i)}. \quad (8)$$

The sequence  $(\delta^{(i)})_{i=1}^{K-1}$  is now examined for periodicity.

*Remark 5.*  $\mathbf{p}_s^{(i)} = \mathbf{p}_s^{(j)}$  for  $i, j \in \mathbb{Z}$  and  $i \neq j$  may occur under specific conditions: For  $\frac{\alpha}{\beta} = 1$  and  $\omega = 1$  both points,  $(0, 1)$  and  $(1, 0)$ , are projected to  $(0.5, 0.5)$ . Therefore  $(\delta^{(i)})_{i=1}^{K-1}$  is not discrete.

### 3 Periodicity

We consider the finite sequence  $(e^{(i)})_{i=1}^L$  with  $L \in \mathbb{N}_{>1}$  to be periodic, if there exists a  $\tilde{\lambda} \in [1, L/2]$  with

$$e^{(i+\tilde{\lambda})} = e^{(i)} \quad \forall i \in [1, L - (L \bmod \tilde{\lambda})]. \quad (9)$$

The period length  $\lambda$  is defined as

$$\lambda := \min \left( \tilde{\lambda}^{(i)} \right). \quad (10)$$

*Proof.*  $\lambda_{\omega=0} = 1$ : As sorting is not necessary, we have

$$\mathbf{p}_s^{(i)} = \mathbf{p}^{(i)} = i \begin{pmatrix} \beta \\ \alpha \end{pmatrix}. \quad (11)$$

According to equation (8), we get the constants

$$\delta^{(i)} = \beta. \quad (12)$$

Hence  $\lambda_{\omega=0} = 1$ . □

## 4 Conjectures

In very rare cases, we see that our C software is not able to calculate a finite period length because the size of the array containing the differences is not sufficient. The array used has 8,000,000,000 64 bit elements and a size of approximately 60 GB.

### 4.1 Conjectures 1 to 5

Using the C software, we were able to support all conjectures listed in the abstract.

### 4.2 Conjecture 6

We found conjecture 6 by providing *OpenAI ChatGPT o3-mini-high* data for  $\omega > 0$  created by the C function `create_test_data` and telling it, it should use *XGBoost* to predict  $\lambda$ . It created the Python function `xgboost` (see [6]).

When we sent the result of this computation, *OpenAI ChatGPT o3-mini-high* wrote of an "indication that the four predictors  $o_n, o_d, a_n, a_d$  [ $\omega = \frac{o_n}{o_d}$ ,  $\alpha = a_n$  and  $\beta = a_d$ ] are in a highly complex, non-linear relationship, which *XGBoost* captures excellently". Then we reported on conjecture 3 and told *OpenAI ChatGPT o3-mini-high*, that we want to find a formula for  $\lambda$ .

Among other suggestions, it proposed the following:

$$\lambda = \omega \Lambda_{\alpha, \beta} \quad (13)$$

By experimenting with the C software we found that

$$\lambda = \lfloor \omega \Lambda_{\alpha, \beta} \rfloor \tag{14}$$

gives even better results. The lowest value we found for  $R^2$  with the C function `test_conjectures` was 0.987994.

## Acknowledgements

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In addition, the *DeepL Translator* (see [5]) has been used to improve the writing style.

## References

- [1] *Page for Yves Meyer at the Abel Prize Homepage*, available at: <https://abelprize.no/abel-prize-laureates/2017>, accessed on December 15, 2024.
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- [4] OpenAI, *ChatGPT*, available at <https://chat.openai.com>, accessed regularly between October 2024 and April 2025.
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- [6] Dirk Kunert, 2025, *Repository cut-and-project*, available at: <https://github.com/dkunert/cut-and-project>.