Conjectures on the Period Lengths of One-Dimensional Cut-and-Project Sequences

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Abstract

With $\alpha, \beta \in \mathbb{N}$, $\alpha \perp \beta$, $x, y \in \mathbb{R}$, $\omega \in \mathbb{R}_{\geq 0}$ and $i \in \mathbb{Z}$, we consider the points

$$\mathbf{C} = \left(\begin{pmatrix} x \\ y \end{pmatrix} \middle| \; x \in \mathbb{Z},\, y \in \left[\frac{\alpha}{\beta}(x-\omega),\; \frac{\alpha}{\beta}x + \omega \right] \cap \mathbb{Z} \right) \tag{1}$$

to be projected vertically onto $f(x) = \frac{\alpha}{\beta}x$ and measure their euclidian distances $\left(d^{(i)}\right)$. With

$$\Lambda_{\alpha,\beta} \;:=\; \alpha + \beta \;+\; 1,$$

we propose the following conjectures concerning the period length λ of $\left(d^{(i)}\right)$:

1. There is always a finite λ.
2. If $\omega \in (0,1)$, $\lambda_{\alpha,\beta} < \Lambda_{\alpha,\beta}$.
3. If $\omega = 1$, $\lambda_{\alpha,\beta} = \Lambda_{\alpha,\beta}$.
4. If $\omega \in (1,2)$, $\lambda_{\alpha,\beta} \geq \Lambda_{\alpha,\beta}$.
5. If $\omega > 2$, $\lambda_{\alpha,\beta} > \Lambda_{\alpha,\beta}$.
6. If $\omega > 0$, $\lambda_{\alpha,\beta} \approx \lfloor \omega \, \Lambda_{\alpha,\beta} \rfloor$.

We use computational methods to support these conjectures (see [6]).

Remark 1. We will show, that $\lambda_{\omega=0} = 1$.

Remark 2. Conjecture 6 is the result of a conversation with *OpenAI ChatGPT o3-mini-high* (see [4]).

1 Introduction

In 2017, Yves Meyer (École Normale Supérieure Paris-Saclay, France) won the Abel Prize “for his pivotal role in the development of the mathematical theory of wavelets” (see [1]). Terence Tao (University of California, Los Angeles) held the announcement (see [2]) and presented Figure 1. With regard to the distances between the projected points on the right side, he noted the following: ”They repeat themselves, but not in a regular fashion.”

These one-dimensional cut-and-project sequences are explored here.

2 Construction of the Sequences

2.1 Projection

In the figures 1 and 2, the functions that are projected vertically onto are $f(x) = \frac{\alpha}{\beta}x$, the upper, passing through $(0, \omega)$, are given by $u(x) = \frac{\alpha}{\beta}x + \omega$, and the lower, passing through $(\omega, 0)$, by $l(x) = \frac{\alpha}{\beta}(x - \omega)$, with $x \in \mathbb{R}$, $\alpha, \beta \in \mathbb{N}$, and the offsets $\omega \in \mathbb{R}_{\geq 0}$.

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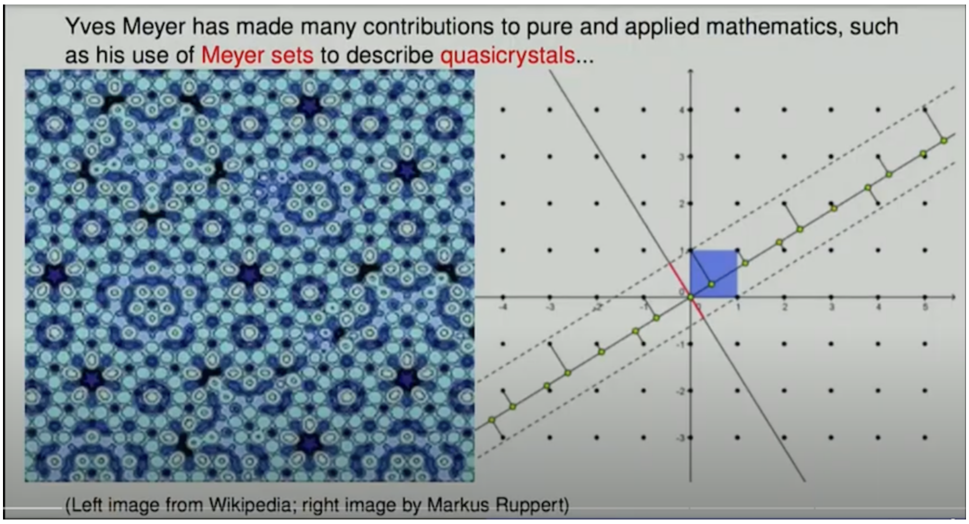

Figure 1: Screenshot from Terence Tao's presentation taken 60 seconds after the video's start (see [2])

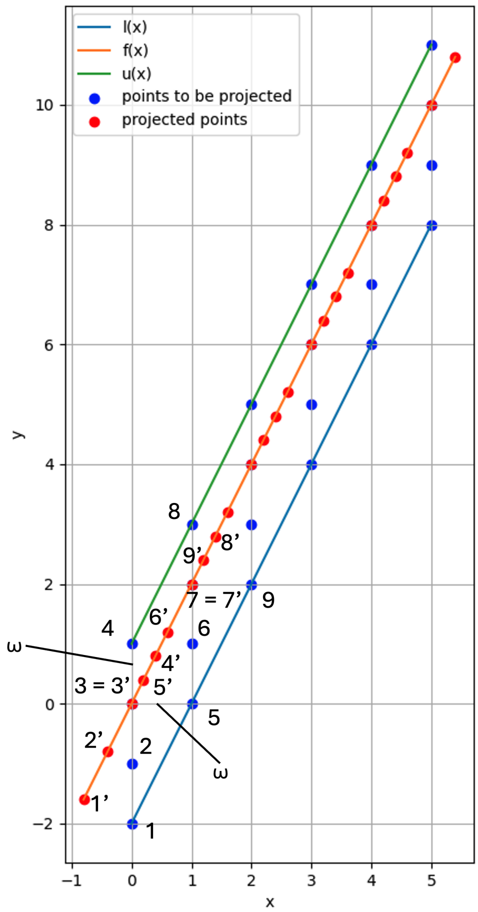

Figure 2: Illustration of the projection of points for $\frac{\alpha}{\beta} = 2$ and $\omega = 1$ within the interval $[0, 5]$

Projecting the points $\mathbf{C}$ described in equation 1 vertically onto $f(x) = \frac{\alpha}{\beta}x$, we get

$$\mathbf{P} = \frac{1}{\alpha^2 + \beta^2} \begin{pmatrix} \beta^2 & \alpha\beta \\ \alpha\beta & \alpha^2 \end{pmatrix} \mathbf{C}. \tag{2}$$

Figure 2 shows that the first projected points are $1'$, $2'$, $3'$, $5'$, $4'$, $6'$, $7'$, $9'$, and $8'$. They are not in the correct order. So, $\mathbf{P}$ must be sorted by the x-coordinates of its points. We get

$$\mathbf{P}_s = \left(\mathbf{p}_s^{(i)}\right), \tag{3}$$

with $i \in \mathbb{Z}$.

Remark 3. We use $\mathbb{Z}$ instead of $\mathbb{N}$, because we have a "two-way infinite sequence" (see [3], p. 106).

Remark 4. Computational experiments show, that sorting has no influence on λ (see C function `test_sorting` in [6]).

For the analysis we limit x to $[0, x_{max}]$, with $x_{max} \in \mathbb{N}$. Examining Figure 2 again, we observe the following squared distances:

Index	<u>1</u>	2	3	4	5	...	18	19	20	21	22	<u>23</u>
Value	0.8	0.8	0.2	1.48	0.2	...	0.8	0.2	1.48	0.2	0.8	0.8

Table 1: Squared distances in Figure 2 for $x_{max} = 5$

By selecting the interval $[0, 5]$, 0.8 appears at index 1 and index 23, making the sequence appear non-periodic despite the repetition of $(0.8, 0.2, 1.48, 0.2)$. However, if we choose the interval $[0, 10]$, the sequence continues as expected after index 22:

Index	...	22	<u>23</u>	24	25	...
Value	...	0.8	0.2	1.48	0.2	...

Table 2: Squared distances in Figure 2 for $x_{max} = 10$

To solve this problem in the computational experiments, we delete elements from the beginning and the end of $\mathbf{P}_s$, while no finite λ is found. We limit the number of removed elements to 10% of the sequence length. This way we get K projected points.

2.2 Distance Calculation

The euclidian distances of the ordered projected points are

$$d^{(i)} = \left\| \mathbf{p}_s^{(i+1)} - \mathbf{p}_s^{(i)} \right\|. \tag{4}$$

Every injective function applied to the elements of $\left(d^{(i)}\right)_{i=1}^{K-1}$ does not change λ. $f(x) = x^2$ is injective for $x \in \mathbb{R}_{\geq 0}$. Thus, we can use the squared distances $\left(s^{(i)}\right)_{i=1}^{K-1}$ instead of the euclidian ones:

$$\begin{aligned} s^{(i)} &= \left(p_{xs}^{(i+1)} - p_{xs}^{(i)}\right)^2 + \left(p_{ys}^{(i+1)} - p_{ys}^{(i)}\right)^2 & (5) \\ &= \left(1 + \frac{\alpha^2}{\beta^2}\right)\left(p_{xs}^{(i+1)} - p_{xs}^{(i)}\right)^2, & (6) \end{aligned}$$

because

$$p_{ys}^{(i)} = \frac{\alpha}{\beta} p_{xs}^{(i)}. \tag{7}$$

$f(x) = \gamma x$ and $f(x) = \sqrt{x}$ are injective for $\gamma \in \mathbb{R}_{\neq 0}$ and $x \in \mathbb{R}_{\geq 0}$ as well. We get

$$\tilde{\delta}^{(i)} = p_{xs}^{(i+1)} - p_{xs}^{(i)}. \tag{8}$$

Equation (2) gives

$$p_x^{(i)} = \frac{\beta}{\alpha^2 + \beta^2} \left(\beta c_x^{(i)} + \alpha c_y^{(i)} \right), \tag{9}$$

where we can omit the constant $\frac{\beta}{\alpha^2+\beta^2}$:

$$\tilde{p}_x^{(i)} = \beta c_x^{(i)} + \alpha c_y^{(i)}. \tag{10}$$

When we sort $\left(\tilde{p}_x^{(i)}\right)$, the result is

$$\delta^{(i)} = \tilde{p}_{xs}^{(i+1)} - \tilde{p}_{xs}^{(i)}. \tag{11}$$

$\left(\delta^{(i)}\right)_{i=1}^{K-1}$ is now examined for periodicity.

Remark 5. $\mathbf{p}_s^{(i)} = \mathbf{p}_s^{(j)}$ for $i, j \in \mathbb{Z}$ and $i \neq j$ may occur under specific conditions: For $\frac{\alpha}{\beta} = 1$ and $\omega = 1$ both points, $(0, 1)$ and $(1, 0)$, are projected to $(0.5, 0.5)$. Therefore $\left(\delta^{(i)}\right)_{i=1}^{K-1}$ is not discrete.

3 Periodicity

We consider the finite sequence $\left(e^{(i)}\right)_{i=1}^{L}$ with $L \in \mathbb{N}_{>1}$ to be periodic, if there exists a $\tilde{\lambda} \in [1, L//2]$ with

$$e^{(i+\tilde{\lambda})} = e^{(i)} \quad \forall i \in [1, L - (L \bmod \tilde{\lambda})]. \tag{12}$$

The period length λ is defined as

$$\lambda := \min\left(\tilde{\lambda}^{(i)}\right). \tag{13}$$

Proof. $\lambda_{\omega=0} = 1$: As sorting is not necessary, we have

$$\mathbf{p}_s^{(i)} = \mathbf{p}^{(i)} = i \begin{pmatrix} \beta \\ \alpha \end{pmatrix}. \tag{14}$$

If we use equation (8), we obtain the constants

$$\tilde{\delta}^{(i)} = \beta. \tag{15}$$

Hence, $\lambda_{\omega=0} = 1$. □

4 Conjectures

In very rare cases, our C software fails to compute a finite period length because the array holding the differences—despite containing 8,000,000,000 64-bit elements (approximately 60 GB)—is not sufficiently large.

4.1 Conjectures 1 to 5

Using the C software, we were able to support all conjectures listed in the abstract.

4.2 Conjecture 6

We found conjecture 6 by providing *OpenAI ChatGPT o3-mini-high* data for $\omega > 0$ created by the C function `create_test_data` and telling it, it should use *XGBoost* to predict λ. It created the Python function `xgboost` (see [6]).

When we sent the result of this computation, *OpenAI ChatGPT o3-mini-high* wrote of an "indication that the four predictors o_n, o_d, a_n, a_d [$\omega = \frac{o_n}{o_d}$, $\alpha = a_n$ and $\beta = a_d$] are in a highly complex, non-linear relationship, which *XGBoost* captures excellently". Then we reported on conjecture 3 and told *OpenAI ChatGPT o3-mini-high*, that we want to find a formula for λ.

Among other suggestions, it proposed the following:

$$\lambda = \omega \ \Lambda_{\alpha,\beta} \tag{16}$$

By experimenting with the C software we found that

$$\lambda = \lfloor \omega \ \Lambda_{\alpha,\beta} \rfloor \tag{17}$$

gives even better results. The lowest value we found for R^2 with the C function `test_conjectures` was 0.987994.

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In addition, the *DeepL Translator* (see [5]) has been used to improve the writing style.

References

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