Conjectures on the Period Lengths of One-Dimensional Cut-and-Project Sequences

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Abstract

With $\alpha, \beta \in \mathbb{N}$, $\alpha \perp \beta$, $x, y \in \mathbb{R}$, $\omega \in \mathbb{R}_{\geq 0}$ and $i \in \mathbb{Z}$, we consider the points

$$\mathbf{P} = \left(\begin{pmatrix} x \\ y \end{pmatrix} \middle| \, x \in \mathbb{Z}, \, y \in \left[\frac{\alpha}{\beta}(x - \omega), \, \frac{\alpha}{\beta}x + \omega \right] \cap \mathbb{Z} \right) \tag{1}$$

projected vertically onto $f(x) = \frac{\alpha}{\beta}x$ and measure their euclidian distances $\left(d^{(i)}\right)$. With

$$\Lambda_{\alpha,\beta} \; := \; \alpha + \beta \; + \; 1,$$

we propose the following conjectures concerning the period length λ of $\left(d^{(i)}\right)$:

1. There is always a finite λ.
2. If $\omega \in (0, 1)$, $\lambda_{\alpha,\beta} < \Lambda_{\alpha,\beta}$
3. If $\omega = 1$, $\lambda_{\alpha,\beta} = \Lambda_{\alpha,\beta}$.
4. If $\omega \in (1, 2)$, $\lambda_{\alpha,\beta} \geq \Lambda_{\alpha,\beta}$
5. If $\omega > 2$, $\lambda_{\alpha,\beta} > \Lambda_{\alpha,\beta}$
6. If $\omega \neq 1$, $\lambda \approx \lfloor (\omega \, \Lambda_{\alpha,\beta} \rfloor$

We use numerical methods to support these conjectures.

Remark 1. We will show, that $\lambda_{\omega=0} = 1$.

Remark 2. Conjecture 6 is the result of a conversation with *ChatGPT o3-mini-high*.

1 Introduction

In 2017, Yves Meyer (École Normale Supérieure Paris-Saclay, France) won the Abel Prize “for his pivotal role in the development of the mathematical theory of wavelets” (see [1]). Terence Tao (University of California, Los Angeles) held the announcement (see [2]) and presented Figure 1.

Observing the distances of the points (see Figure 1, right), he stated the following: ”They repeat themselves, but not in a regular fashion.”

These one-dimensional cut-and-project sequences are explored here. C and Python software is used to support our conjectures (see [4]).

2 Construction of the Set

2.1 Projection

In Figure 2, the function that is projected vertically onto is $f(x) = \frac{\alpha}{\beta}x$, the upper, passing through $(0, \omega)$, is given by $u(x) = \frac{\alpha}{\beta}x + \omega$, and the lower, passing through $(\omega, 0)$, by $l(x) = \frac{\alpha}{\beta}(x - \omega)$, with $x \in \mathbb{R}$, $\alpha, \beta \in \mathbb{N}$, and the offset $\omega \in \mathbb{R}_{\geq 0}$.

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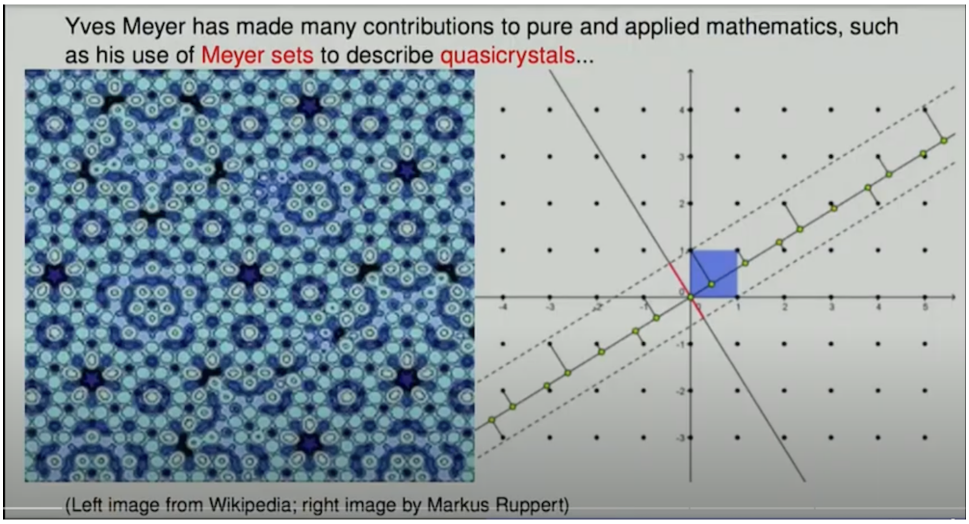

Figure 1: Screenshot from Terence Tao's presentation taken 60 seconds after the video's start

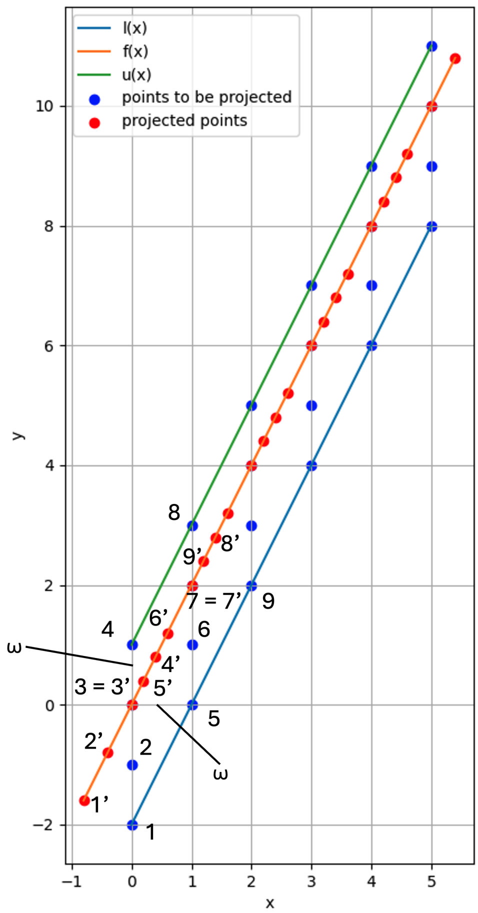

Figure 2: Illustration of the projection of points for $\frac{\alpha}{\beta} = 2$ and $\omega = 1$ within the interval $[0, 5]$

Projecting the points shown in equation 1 vertically on $f(x) = \frac{\alpha}{\beta}x$, we get

$$\tilde{\mathbf{P}} = \frac{1}{\alpha^2 + \beta^2} \begin{pmatrix} \beta^2 & \alpha\beta \\ \alpha\beta & \alpha^2 \end{pmatrix} \mathbf{P}. \tag{2}$$

Figure 2 shows that the first projected points are $1'$, $2'$, $3'$, $5'$, $4'$, $6'$, $7'$, $9'$, and $8'$. They are not in the correct order. So, $\tilde{\mathbf{P}}$ must be sorted by the x-coordinates of its points. We get

$$\tilde{\mathbf{P}}_s = \left(\tilde{\mathbf{p}}_s^{(i)}\right), \tag{3}$$

with $i \in \mathbb{Z}$.

Remark 3. We use $\mathbb{Z}$ instead of $\mathbb{N}$, because we have a "two-way infinite sequence" (see [3], p. 106).

Remark 4. Numerical experiments show, that sorting has no influence on λ.

For the numerical analysis we limit x to $[0, x_{max}]$, with $x_{max} \in \mathbb{N}$. Examining Figure 2 again, we observe the following squared distances:

Index	1	2	3	4	5	...	18	19	20	21	22	23
Value	0.8	0.8	0.2	1.48	0.2	...	0.8	0.2	1.48	0.2	0.8	0.8

Table 1: Squared distances in Figure 2 for $x_{max} = 5$

By choosing the interval $[0, 5]$, 0.8 is added at the beginning and at the end, making the sequence appear non-periodic despite the repetition of $(0.8, 0.2, 1.48, 0.2)$. Indeed, if we choose the interval $[0, 10]$, the series continues as expected:

Index	...	22	23	24	25	...
Value	...	0.8	0.2	1.48	0.2	...

Table 2: Squared distances in Figure 2 for $x_{max} = 10$

To solve this problem in the numerical experiments, we delete elements from the beginning and the end of P_s, while no finite λ is found. We limit the number of removed elements to 10% of the sequence length. This way we get K projected points.

2.2 Distance Calculation

The following applies to the euclidian distances of the ordered projected points with $\tilde{p}_{xs}^{(i)} \in [0, x_{max}]$:

$$\left(d^{(i)}\right)_{i=1}^{K-1} = \left\|\tilde{\mathbf{p}}_s^{(i+1)} - \tilde{\mathbf{p}}_s^{(i)}\right\| \tag{4}$$

Every injective function applied to the elements of $\left(d^{(i)}\right)_{i=1}^{K-1}$ does not change λ. $f(x) = x^2$ is injective for $x \in \mathbb{R}_{\geq 0}$. Thus, we can use the squared distances $\left(s^{(i)}\right)_{i=1}^{K-1}$ instead of the euclidian ones:

$$s^{(i)} = \left(\tilde{p}_{xs}^{(i+1)} - \tilde{p}_{xs}^{(i)}\right)^2 + \left(\tilde{p}_{ys}^{(i+1)} - \tilde{p}_{ys}^{(i)}\right)^2 \tag{5}$$

$$= \left(1 + \frac{\alpha^2}{\beta^2}\right)\left(\tilde{p}_{xs}^{(i+1)} - \tilde{p}_{xs}^{(i)}\right)^2, \tag{6}$$

because

$$\tilde{p}_{ys}^{(i)} = \frac{\alpha}{\beta}\tilde{p}_{xs}^{(i)}. \tag{7}$$

$f(x) = \gamma x$ and $f(x) = \sqrt{x}$ are injective for $\gamma \in \mathbb{R}_{\neq 0}$ and $x \in \mathbb{R}_{\geq 0}$ as well. We get

$$\tilde{s}^{(i)} = \left(\tilde{p}_{xs}^{(i+1)} - \tilde{p}_{xs}^{(i)}\right). \tag{8}$$

From equation (2) it follows that

$$\tilde{p}_{xs}^{(i)} = \frac{\beta}{\alpha^2 + \beta^2}\left(\beta p_{xs}^{(i)} + \alpha p_{ys}^{(i)}\right). \tag{9}$$

Here we also can omit the constant $\frac{\beta}{\alpha^2+\beta^2}$. Finally, we obtain

$$\left(\delta^{(i)}\right)_{i=1}^{K-1} = \left(\beta\left(p_{xs}^{(i+1)} - p_{xs}^{(i+1)}\right) + \alpha\left(p_{ys}^{(i+1)} - p_{ys}^{(i+1)}\right)\right)_{i=1}^{K-1}. \tag{10}$$

This sequence is now examined for periodicity.

Remark 5. $\tilde{p}_{xs}^{(i)} = \tilde{p}_{xs}^{(j)}$ for $i, j \in \mathbb{Z}$ and $i \neq j$ may occur under specific conditions: For $\frac{\alpha}{\beta} = 1$ and $\omega = 1$ both points, $(0,1)$ and $(1,0)$, are projected to $(0.5, 0.5)$. Therefore $\left(\delta^{(i)}\right)_{i=1}^{K-1}$ is not discrete.

3 Periodicity

We consider the finite sequence $\left(e^{(i)}\right)_{i=1}^{L}$ with $L \in \mathbb{N}_{>1}$ to be periodic, if there exists a $\tilde{\lambda} \in [1, L//2]$ with

$$e^{(i+\tilde{\lambda})} = e^{(i)} \; \forall i \in [1, L - \tilde{\lambda}]. \tag{11}$$

The period length λ is defined as

$$\lambda := \min\left(\tilde{\lambda}^{(i)}\right). \tag{12}$$

Proof. $\lambda_{\omega=0} = 1$: As sorting is not necessary, $\mathbf{P}_s = \mathbf{P}$. Thus,

$$\left(\delta^{(i)}\right)_{i=1}^{K-1} = \left(\beta\left(p_x^{(i+1)} - p_x^{(i)}\right) + \alpha\left(p_y^{(i+1)} - p_y^{(i)}\right)\right)_{i=1}^{K-1} \tag{13}$$

according to 10. As $p_x^{(i+1)} - p_x^{(i)} = 1$ and $p_y^{(i+1)} - p_y^{(i)} = \frac{\alpha}{\beta}$, we get

$$\left(\delta^{(i)}\right)_{i=1}^{K-1} = \left(\beta + \frac{\alpha^2}{\beta}\right)_{i=1}^{K-1}, \tag{14}$$

which consists of constant elements. As $e^{(i+1)} = e^{(i)} \; \forall i \in [1, K-2]$, $\lambda_{\omega=0} = 1$. $\square$

4 Conjectures

In very rare cases, we see that our C software is not able to calculate a finite period length because the size of the array containing the differences is not sufficient. This array consists of 8,000,000,000 elements with 64 bit and has a size of more than 60 GB.

4.1 Conjectures 1 to 5

Using the C software, we were able to support all conjectures listed in the abstract.

Remark 6. The fraction for $a = \sqrt{2}$, represented with 64 bit floating-point arithmetic, is

$$\tilde{a} = \frac{6{,}369{,}051{,}672{,}525{,}773}{4{,}503{,}599{,}627{,}370{,}496}. \tag{15}$$

If conjecture 3 is correct, we would obtain

$$\lambda_{a=\tilde{a},\,\omega=1} = 10{,}872{,}651{,}299{,}896{,}270. \tag{16}$$

Such a sequence with 8 bit per element would take up around 11 PB of memory. Provided conjecture 3 was correct, there is no finite λ, because $\sqrt{2}$ can only be represented by infinitely large numerators and denominators.

4.2 Conjecture 6

We found conjecture 6 by providing *ChatGPT o3-mini-high* data created by the C function `create_test_data` and telling it, it should use *XGBoost* to predict λ. It created the Python function `xgboost` (see [4]).

When we sent ChatGPT the result of this computation, it wrote of an "indication that the four predictors o_n, o_d, a_n, a_d [$\omega = \frac{o_n}{o_d}$, $\alpha = a_n$ and $\beta = a_d$] are in a highly complex, non-linear relationship, which XGBoost captures excellently". Then we reported on conjecture 3 and told *ChatGPT o3-mini-high*, that we want to find a formula for λ.

Among other suggestions, it proposed the following:

$$\lambda = \omega\, \Lambda_{\alpha,\beta} \tag{17}$$

By experimenting with the C software we found that

$$\lambda = \lfloor \omega\, \Lambda_{\alpha,\beta} \rfloor \tag{18}$$

gives even better results. The lowest value we found for R^2 with the C function `test_conjectures` was 0.987994.

References

[1] *Page for Yves Meyer at the Abel Prize Homepage*, available at: `https://abelprize.no/abel-prize-laureates/2017`, accessed on December 15, 2024.

[2] *Terence Tao on Yves Meyer's work on Wavelets*, available at: `https://youtu.be/AnkinNVPjyw`, accessed on December 15, 2024.

[3] Marjorie Senechal, *Quasicrystals and geometry*, 2009. ISBN: 978-0-521-57541-6.

[4] D. Kunert, 2025, *Code for this article*, Available at `https://github.com/dkunert/cut-and-project/tree/main/src`.

5 Conclusion

TODO: Missing!