Exercise 4.3: The EM algorithm

- Main idea is to prove EM-alg. works in

$\log p(X;\Theta_{t+1}) \geq \log p(X;\Theta_t)$, for every iteration.

a) $\log p(x|\Theta_{t+1}) - \log p(x|\Theta_t) > 0$

b) $E_{q_p(z|x;\Theta_t)}[\log p(x|\Theta_{t+1}) - \log p(x|\Theta_t)] > 0$

$\Rightarrow$ As Z does not appear in the difference

$\Rightarrow$ the value is $[\log p(x|\Theta_{t+1}) - \log p(x|\Theta_t)]$

c) $\log p(x|\Theta_{t+1}) - \log p(x|\Theta_t)$

$\Rightarrow \log\left(\frac{p(x|\Theta_{t+1})\cdot p(z|x;\Theta_{t+1})}{p(z|x;\Theta_{t+1})}\right) - \log\left(\frac{p(x|\Theta_t)\cdot p(z|x;\Theta_t)}{p(z|x;\Theta_t)}\right)$

d/

$$E_q\left[\log\left(\frac{p(x|\theta_{t+1})\cdot p(z|x;\theta_{t+1})}{p(z|x;\theta_{t+1})}\right)-\log\left(\frac{p(x|\theta_t)\cdot p(z|x;\theta_t)}{p(z|x;\theta_t)}\right)\right]$$

$$\| \cdot p(z|x;\theta_t)$$

$$\Sigma\left(\log\left(\frac{p(x|\theta_{t+1})\cdot p(z|x;\theta_{t+1})}{p(z|x;\theta_{t+1})}\right)-\log\left(\frac{p(x|\theta_t)\cdot p(z|x;\theta_t)}{p(z|x;\theta_t)}\right)\right)\cdot p(z|x;\theta_t)$$

$$=\underbrace{\Sigma p(z|x;\theta_t)\cdot\log\left(\frac{p(x|\theta_{t+1})\cdot p(z|x;\theta_{t+1})}{p(z|x;\theta_{t+1})}\right)-\Sigma p(z|x;\theta_t)\cdot\log\left(\frac{p(x|\theta_t)\cdot p(z|x;\theta_t)}{p(z|x;\theta_t)}\right)}_{\geq 0}$$

$\Rightarrow$

$$=\Sigma p(z|x;\theta_t)\cdot\log\left(\frac{p(x|\theta_{t+1})\cdot \cancel{p(z|x;\theta_{t+1})}}{\cancel{p(z|x;\theta_{t+1})}}\right)=\Sigma p(z|x;\theta_t)\cdot\log\left(\frac{p(x|\theta_t)\cdot p(z|x;\theta_t)}{\cancel{p(z|x;\theta_t}}\right)$$

$$=\Sigma p(z|x;\theta_t)\log p(z|x;\theta_{t+1})-\Sigma p(z|x;\theta_t)(\log(p(z|\theta_t))$$

$$=Q(\theta_{t+1}|\theta_t)-\Sigma p(z|x;\theta_t)\cdot\log p(z|x,\theta_{t+1})$$

(one side)

$$=Q(\theta_{t+1}|\theta_t)-\Sigma p(z|x;\theta_t)\cdot\log p(z|x;\theta_{t+1})-\ldots$$

$$\ldots(Q(\theta_t|\theta_t)-\Sigma p(z|x;\theta_t)\cdot\log p(z|x;\theta_t))$$

$$=Q(\theta_{t+1}|\theta_t)-Q(\theta_t|\theta_t)-\Sigma p(z|x;\theta_t)\log p(z|x;\theta_{t+1})+\ldots$$

$$\Sigma p(z|x;\theta_t)\cdot\log(p(z|x;\theta_t)$$

taken from last equation

$a = \sum p(z|x;\theta_t) \cdot (\log p(z|x;\theta_t) - \log p(z|x;\theta_{t+1}))$

$\Rightarrow$ Using Hint $\log(a/b) = \log(a) - \log(b)$

$\Rightarrow \sum p(z|x;\theta_t) \cdot \log \frac{p(z|x;\theta_t)}{p(z|x;\theta_{t+1})}$

e)

$Q(\theta_{t+1}|\theta_t) - Q(\theta_t|\theta_t) + \sum p(z|x;\theta_t) \cdot \log \frac{p(z|x;\theta_t)}{p(z|x;\theta_{t+1})}$

$Q(\theta_{t+1}|\theta_t) - Q(\theta_t|\theta_t) + \underbrace{\sum p(z|x;\theta_t) \cdot \log \frac{p(z|x;\theta_t)}{p(z|x;\theta_{t+1})}}_{\geq 0} =$

$\log p(x|\theta_{t+1}) - \log p(x|\theta_t)$

Hence $\Rightarrow$

$\log p(x|\theta_{t+1}) - \log p(x|\theta_t) = Q(\theta_{t+1}|\theta_t) - Q(\theta_t|\theta_t)$

$\Rightarrow$

$\log p(x|\theta_{t+1}) - \log p(x|\theta_t) - Q(\theta_{t+1}|\theta_t) - Q(\theta_t|\theta_t) \geq 0$

from m-step

$$\theta_{t+1} = \text{argmax } Q(\theta|\theta_t)$$

Hence

$$\Rightarrow Q(\theta_{t+1}|\theta_t) \geq Q(\theta_t|\theta_t)$$

$$\Rightarrow Q(\theta_{t+1}|\theta_t) - Q(\theta_t|\theta_t) \geq 0$$

Therefore,

$$\log p(x|\theta_{t+1}) - \log p(x|\theta_t) \geq 0$$