Data Stat. 840
Exercises 7

7.1 The forward backward algorithm

$p(z_1)=0{,}2$, $N=7$, $N_S=5$
$x=[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$ = the, quick, fox, jumps, over, a, dog

$$p(x_1, x_2 ..., x_7) = \sum_{k=1}^{5} p_f(t,k)\, p_b(t,k)$$

Initialization: $t=4$
Lets go through all the k values.

$k=1$
$$p_f(4,1) = \sum_{j=1}^{5} p_f(3,j)\cdot\theta_{1|j}\cdot\underline{\beta_1(x_4)} = 0 \Rightarrow \beta_1(x_4)=0$$
$$p_b(4,1) = \sum_{j=1}^{5} p_b(5,j)\cdot\theta_{j|1}\cdot\beta_j(x_5) = 0$$

$k=2$
$$p_f(4,2) = \sum_{j=1}^{5} p_f(3,j)\,\theta_{2|j}\,\beta_2(x_4) = 0$$
$$p_b(4,2)\ \sum_{j=1}^{5} p_b(5,j)\cdot\theta_{j|2}\,\beta_j(x_5) = 0$$

$k=3$
$$p_f(4,3) = \sum_{j=1}^{5} p_f(3,j)\,\theta_{3|j}\,\beta_3(x_4) = 0$$
$$p_b(4,3) = \sum_{j=1}^{5} p_b(5,j)\cdot\theta_{j|3}\cdot\beta_j(x_5) = 0$$

$k=4$
$$p_f(4,4) = \sum_{j=1}^{5} p_f(3,j)\,\theta_{4|j}\cdot\beta_4(x_4) = p_f(3,5)\,\theta_{4|5}\,\beta_4(x_4)$$
$$p_b(4,4) = \sum_{j=1}^{5} p_b(5,j)\cdot\theta_{j|4}\cdot\beta_j(x_5) = p_b(5,2)\,\theta_{2|4}\,\beta_2(x_5)$$

$k=5$
$$p_f(4,5) = \sum_{j=1}^{5} p_f(3,j)\,\theta_{5|j}\cdot\beta_5(x_4) = 0$$
$$p_b(4,5) = \sum_{j=1}^{5} p_b(5,j)\,\theta_{j|5}\cdot\beta_j(x_5) = p_b(5,2)\,\theta_{2|5}\,\beta_2(x_5)$$

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Hence, the probabilities

$$P(x_1, x_2 \dots, x_7) = \sum_{k=1}^{5} p_f(4,k)\, p_b(4,k)$$

$$= p_f(4,4) \cdot p_b(4,4)$$

$$= p_f(3,5)\,\theta_{4|5}\,\beta_4(x_4) \cdot p_b(5,2)\,\theta_{2|4}\,\beta_2(x_5)$$

$$= \underline{p_f(3,5)} \cdot 0{,}5 \cdot 0{,}1 \cdot \underline{p_b(5,2)} \cdot 1 \cdot 0{,}2$$

Then we will use recursion

$$p_f(3,5) = \sum_{j=1}^{5} p_f(2,j)\,\theta_{5|j}\,\beta_5(x_3)$$

$$= p_f(2,1)\,\theta_{5|1}\,\beta_5(x_3) + p_f(2,3)\,\theta_{5|3}\,\beta_5(x_3)$$

$$= \underline{p_f(2,1)} \cdot 0{,}5 \cdot 0{,}3 + \underline{p_f(2,3)} \cdot 0{,}7 \cdot 0{,}3$$

$$p_b(5,2) = \sum_{j=1}^{5} p_b(6,j)\,\theta_{j|2}\,\beta_j(x_6) = p_b(6,1)\,\theta_{1|2}\,\beta_1(x_6)$$

$$= \underline{p_b(6,1)} \cdot 1 \cdot 0{,}6$$

Recursion again:

$$p_f(2,1) = \sum_{j=1}^{5} p_f(1,j)\,\theta_{1|j}\,\beta_1(x_2) = 0$$

$$p_f(2,3) = \sum_{j=1}^{5} p_f(1,j)\,\theta_{3|j}\,\beta_3(x_2) = p_f(1,1) \cdot 0{,}5 \cdot 0{,}5 + p_f(1,3) \cdot 0{,}25 \cdot 0{,}5$$

$$p_b(6,1) = (7,5)\,\theta_{5|1}\,\beta_5(x_7)$$

$\Rightarrow$ $p_f(1,1) = \beta_1(x_1)\pi_1 = 0{,}4 \cdot 0{,}2 = 0{,}08$ $\Rightarrow 0{,}08 \cdot 0{,}5 \cdot 0{,}5 + 0 = 0{,}02 = p_f(2,3)$

$0 = p_f(2,1)$

$p_f(1,3) = \underbrace{\beta_3(x_1)}_{0}\pi_3 = 0$ $\Rightarrow$ $\Rightarrow p_f(2,1) \cdot 0{,}5 \cdot 0{,}3 + p_f(2,3) \cdot 0{,}7 \cdot 0{,}3$

$0 + 0{,}02 \cdot 0{,}7 \cdot 0{,}3 = 0{,}0042$

$p_b(7,5) = 1?$

$p_f(3,5) = 0{,}0042$

$0{,}0042 \cdot 0{,}5 \cdot 0{,}1 = 0{,}00021$

$0{,}00021 \cdot p_b(4,4)$

$0{,}00021 \cdot 0{,}0036$

$= \underline{0{,}000000756}$

7.2 The Viterbi algorithm

$P(z_1) = 0{,}2$, $N = 8$, $N_S = 5$
$x = [x_1, x_2, \ldots, x_8] \neq$ you, claim you can book a round hotel

$z_t = $ P_max(timestep t)

$t = 1$
$Best(1,1) = \pi_1 \beta_1(x_1) = 0{,}2 \cdot 0{,}1 = 0{,}02$
$Best(1,2) = 0{,}2 \cdot 0{,}3 = 0{,}06$
$Best(1,3) = 0{,}2 \cdot 0 = 0$
$Best(1,4) = 0{,}2 \cdot 0 = 0$
$Best(1,5) = 0{,}2 \cdot 0{,}05 = 0{,}001$

$t = 2$
$Best(2,1) = \max_j Best(1,j)\, \theta_{1|j}\, \beta_1(x_2) = 0$
$Best(2,2) = \max_j Best(1,j)\, \theta_{2|j}\, \beta_2(x_2) = 0$
$Best(2,3) = \max_j Best(1,j)\, \theta_{3|j}\, \beta_3(x_2)$
$= Best(1,1)\, \theta_{3|1} \cdot 0{,}1$
$= 0{,}2 \cdot 0{,}3 \cdot 0{,}1 = 0{,}0006$

$BestZ(2,3) = 1$
$Best(2,4) = \max_j Best(1,j)\, \theta_{4|j}\, \beta_4(x_2) = 0$
$Best(2,5) = \max_j Best(1,j)\, \theta_{5|j}\, \beta_5(x_2)$
$= Best(1,2)\, \theta_{5|2} \cdot 0{,}15$
$= 0{,}06 \cdot 0{,}6 \cdot 0{,}15$
$= 0{,}00054$

$BestZ(2,5) = 2$

$t = 3$
$Best(3,1) = \max_j Best(2,j)\, \theta_{1|j}\, \beta_1(x_3) = 0$
$Best(3,2) = \max_j Best(2,j)\, \theta_{2|j}\, \beta_2(x_3)$
$= Best(2,3)\, \theta_{2|3} \cdot 0{,}3$
$= 0{,}0006 \cdot 0{,}75 \cdot 0{,}3 = 0{,}000135 \quad \Rightarrow BestZ(3,2) = 3$

$Best(3,3) = \max_j Best(2,j)\, \theta_{3|j}\, \beta_3(x_3) = 0$
$Best(3,4) = \max_j Best(2,j)\, \theta_{4|j}\, \beta_4(x_3) = 0$
$Best(3,5) = \max_j Best(2,j)\, \theta_{5|j}\, \beta_5(x_3) = 0$

$t=4$

$\text{Best}(4,1) = \max_j \text{Best}(3,j)\, \theta_{1|j}\, \beta_1(x_4) = 0$

$\text{Best}(4,2) = \max_j \text{Best}(3,j)\, \theta_{2|j}\, \beta_2(x_4) = 0$

$\text{Best}(4,3) = \max_j \text{Best}(3,j)\, \theta_{3|j}\, \beta_3(x_4) = 0$

$\text{Best}(4,4) = \max_j \text{Best}(3,j)\, \theta_{4|j}\, \beta_4(x_4) = 0$

$\text{Best}(4,5) = \max_j \text{Best}(3,j)\, \theta_{5|j}\, \beta_5(x_6)$

$= \text{Best}(3,2)\, \theta_{5|2}\, 0{,}05$

$= 0{,}000135 \cdot 0{,}6 \cdot 0{,}05$

$= 0{,}00000405$

$\text{BestZ}(4,5) = 2$

$t=5$

$\text{Best}(5,1) = \max_j \text{Best}(4,j)\, \theta_{1|j}\, \beta_1(x_5) = 0$

$\text{Best}(5,2) = \max_j \text{Best}(4,j)\, \theta_{2|j}\, \beta_2(x_5) = 0$

$\text{Best}(5,3) = \max_j \text{Best}(4,j)\, \theta_{3|j}\, \beta_3(x_5)$

$= \text{Best}(4,5)\, \theta_3 \cdot 0{,}1$

$= 0{,}00000405 \cdot 1 \cdot 0{,}1$

$= 4{,}05 \cdot 10^{-7}$

$\text{BestZ}(5,3) = 5$

$\text{Best}(5,4) = \max_j \text{Best}(4,j)\, \theta_{4|j}\, \beta_4(x_5) = 0$

$\text{Best}(5,5) = \max_j \text{Best}(4,j)\, \theta_{5|j}\, \beta_5(x_5) = 0$

$t=6$

$\text{Best}(6,1) = \max_j \text{Best}(5,j)\, \theta_{1|j}\, \beta_1(x_6) = 0$

$\text{Best}(6,2) = \max_j \text{Best}(5,j)\, \theta_{2|j}\, \beta_2(x_6)$

$= \text{Best}(5,3)\, \theta_{2|3} \cdot 0{,}3$

$= 4{,}05 \cdot 10^{-7} \cdot 0{,}75 \cdot 0{,}3$

$= 9{,}1125 \cdot 10^{-8}$

$\text{BestZ}(6,2) = 3$

$\text{Best}(6,3) = \max_j \text{Best}(5,j)\, \theta_{3|j}\, \beta_3(x_6) = 0$

$\text{Best}(6,4 = \max_j \text{Best}(5,j)\, \theta_{4|j}\, \beta_4(x_6) = 0$

$\text{Best}(6,5) = \max_j \text{Best}(5,j)\, \theta_{5|j}\, \beta_5(x_6) = 0$

$t=7$

$\text{Best}(7,1) = \max_j \text{Best}(6,j)\, \theta_{1|j}\, \beta_1(x_7) = 0$

$\text{Best}(7,2) = \max_j \text{Best}(6,j)\, \theta_{2|j}\, \beta_2(x_7) = 0$

$\text{Best}(7,3 = \max_j \text{Best}(6,j)\, \theta_{3|j}\, \beta_3(x_7) = 0$

$\text{Best}(7,4) = \max_j \text{Best}(6,j)\, \theta_{4|j}\, \beta_4(x_7)$

$= \text{Best}(6,2)\, \theta_{4|2} \cdot 0{,}3 = 9{,}1125 \cdot 10^{-8} \cdot 0{,}4 \cdot 0{,}3 = 1{,}0935 \cdot 10^{-8}$

$\text{BestZ}(7,4) = 2$

$\text{Best}(7,5) = \max_j \text{Best}(6,j)\, \theta_{5|j}\, \beta_5(x_7)$

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$= Best(6,2)\,\theta_{5|2} \cdot 0{,}1$

$= 9{,}1125 \cdot 10^{-8} \cdot 0{,}6 \cdot 0{,}1 = 5{,}4675 \cdot 10^{-9}$

BestZ = 2

$t = 8$

$Best(8,1) = \max_j Best(7,j)\,\theta_{1|j}\,\beta_1(x_8) = 0$

$Best(8,2) = \max_j Best(7,j)\,\theta_{2|j}\,\beta_2(x_8) = 0$

$Best(8,3) = \max_j Best(7,j)\,\theta_{3|j}\,\beta_3(x_8) = 0$

$Best(8,4) = \max_j Best(7,j)\,\theta_{4|j}\,\beta_4(x_8) = 0$

$Best(8,5) = \max_j Best(7,j)\,\theta_{5|j}\,\beta_5(x_8)$

$= Best(6,4)\,\theta_{5|4} \cdot 0{,}15 = 1{,}0935 \cdot 10^{-8} \cdot 1 \cdot 0{,}10$

$= 1{,}0935 \cdot 10^{-9}$

7 has 2 states 7,4 and 7,5 (2+2)

$\Rightarrow$ most likely state.