Markov chains and algorithmic applications: Deploying a 5G Network in a country

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I. Introduction

For the given project task, we present function optimization using Metropolis-Hastings (MH) algorithm. We introduce four base chains for which we conduct comparative analysis. Additionally, we improve the results by adding simulated annealing method and tuning internal parameters. We analyze how the algorithm evolves towards the solution for G1 and G2 datasets. We elect one base chain as the best chain and examine the expected values from the optimal set of cities included in the solution and the optimal function value with respect to λ.

In the remainder of the report, we describe proposed based chains (Section II), we analyze their behavior (Sections III, IV) and suggest the strategy of changing lambda for the network deployment (Section V).

II. Base Chains

The state in the base chains can be identified with a subset of cities that are chosen for the set S. For the initial state, we choose the two most populated cities.

A. Dummy Base chain

A state is any subset of Cities. A state has $|n|+1$ neighbors, where n is number of cities. Each state has a self-loop. The neighbors of the state S are obtained by flipping a bit from P. Any transition has equal probability. The chain is aperiodic and irreducible by construction and it holds that $\psi_{ij} > 0$ iff $\psi_{ji} > 0$.

B. Heat Bath Base Chain

Another proper base chain is one built upon heat bath dynamics which match our task. The idea behind this chain is to change every step an only coordinate of a vector state chosen uniformly at random. We want to sample from the distribution $\pi\left(x\right) = \frac{e^{-\beta f(x)}}{Z},\ x \in S^n$. Let us denote $e^{-\beta f(x)}$ as $g\left(x\right)$, $x = \left(x_1, ..., x_u, ..., x_n\right)$, $y_u = \left(x_1, ..., x_u^{'}, ..., x_n\right)$ then new states are sampled from the distribution

$$\pi\left(x_u^{'}|x_1, ..., x_{u-1}, x_{u+1}, ..., x_n\right) = \frac{g\left(y_u\right)}{\sum\limits_{y_u \in S} g\left(y_u\right)},$$

which allows avoiding computation of Z.

In our case, again, the state space is $\{0,1\}^n$, where n is a number of cities, 0 and 1 denote a given city is taken or not. Therefore, by using formulas from the homework we can obtain:

$$p\left(i,j\right) = \begin{cases} \frac{\sigma(\beta[f(x)-f(y_u)])}{n}, & \text{if } j = \pi\left(y_u\right) \\ & \text{ for } u \in \{1,..,n\} \\ & \text{ and } x_u \neq x_u^{'} \\ \frac{\sum_{u=1}^{n}\sigma(\beta[f(y_u)-f(x)])}{n}, & \text{if } j = i \\ 0, & \text{otherwise,} \end{cases}$$

where σ is a sigmoid function and each state has n neighbors.

In addition, from the homework, we know that this chain posses the detailed balance property which entail ergodicity and, hence, the desired irreducibility and aperiodicity. Moreover, acceptance probabilities in this case are always 1 since $a_{ij} = \min(1, \frac{\pi_j \phi_{ji}}{\pi_i \phi_{ij}}) = \min(1,1) = 1$

C. 2D Random Walk Base chain

Here, we propose two chains. They can be seen as a modification of a 2D Random Walk chain on a finite set of states. We want to inject the physics of a given problem into our Markov chain to reduce the number of states. If we fix an area of antennas placing, then it is reasonable to take all the area's inner cities to maximize the function. Hence, we consider all the inner cities as a state for a given area. Now let us assume we numbered all the cities. Then, given two cities (two index numbers) we can build the circle such that these cities lie on a bound of its diameter. Then our state-space consists from vectors (d_1, d_2), where d_1 and d_2 are indexes of two cities and $d_1 < d_2$, as states (d_1, d_2) and (d_2, d_1) are the same. A representative example of such a chain, which we call 2D Random Walk (2D RW) further, can be seen in Fig. 1. All the transition probabilities between different states are equal $\frac{1}{4}$. Boundary states have self-loops with probabilities such that the probability to transit is 1 in total. As a result, this chain has slightly less than 5000 states comparing to 2^{100} states in the case of Dummy and Heat Bath base chains.

One can notice that even one transition between states drastically changes the function value. The reason is that we change index numbers of bound cities, which has no relation with the geographical places. Therefore, we construct a chain to make it more robust to the above-mentioned issue. It is presented in Fig. 2 and we call it Improved 2D Random Walk (Imp. 2D RW) further. For each state now, we can move to the left, right, top and bottom until the boundary state. Any transition to leave a given state has an equal probability and is obtained individually depending on the number of possible transitions. Boundary states have self-loops with probabilities such that the probability to transit is 1 in total.

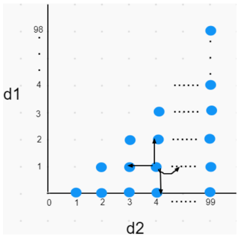

Fig. 1: Transitions from (1,4) (2D)

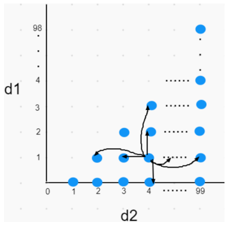

Fig. 2: Transitions from (1,4) (Improv.)

From Fig. 1 and 2 we conclude that chains are irreducible, aperiodic, given self-loops on a bound states. It holds that $\psi_{ij} > 0$ iff $\psi_{ji} > 0$. Therefore, these chains are the proper for Metropolis Hastings algorithm.

D. Optimization

For each state i the objective function is uniquely defined and it is equal to $f(i)$. On a base chain MH algorithm is performed with acceptance probabilities $a_{ij} = \min(1, \frac{\pi_j \psi_{ji}}{\pi_i \psi_{ij}})$, where ψ is transition matrix for a base chain and the ratio $\frac{\pi_j}{\pi_i}$ is calculated as $\frac{e^{\beta f(j)}}{e^{\beta f(i)}}$. We used simulated annealing approach. For each base chain, initial β, number of iterations after β increases (α) and the percentage by which it increases (δ) are tuned and they are depicted in the Table I.

TABLE I: Tuned parameters

Base Chain	G1			G2		
	β	α	δ	β	α	δ
Dummy	1.0	300	1.5	1.0	300	1.25
Heat Bath	1.0	300	1.5	0.55	300	1.5
2D RW	0.1	300	1.0	0.1	300	1.0
Imp. 2D RW	0.55	300	1.0	0.1	300	1.0

III. Analysis on G1

A. Choosing the best base chain and tuning parameters

Figs. 3 and 4 show the dependency of objective function and number of chosen cities on λ parameter correspondingly. β and δ were chosen for each chain and each λ according to the best objective function value. According to Fig. 3, we conclude that all chains behave almost equally with a slight advantage in the Heat Bath base chain case. For the number of chosen cities, in Fig. 4 one can notice similar behavior between all chains, except Dummy one. Therefore, these plots tell us that the Heat Bath base chain is a reasonable choice for our problem.

B. Algorithm behavior through steps

Figures 5-12 depict results for $\lambda = 0.5$. Parameters for simulated annealing are tuned separately for each base chain.

It can be seen (Fig. 5 and Fig. 6) that Dummy and Heat-Bath base chain have stable trend of function increase. In the

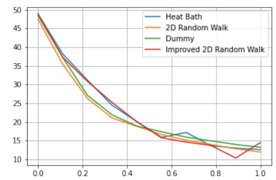

Fig. 3: Function vs. λ

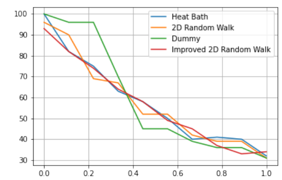

Fig. 4: No. of cities vs. λ

beginning, they tend to choose more and more cities, but the number of chosen cities stabilizes (Fig. 9 and Fig. 10). This is expected since β increases over time.

The last two chains have greatly reduced the number of states compared to Dummy and Base Chain, as stated in II-C. However, for 2D RW chain (Fig. 7 and Fig. 11), the both trends are unstable. On the other, an Imp. 2D RW base chain tends to accept more promising states since it is more robust than 2D RW. On Fig. 8 we can see that high function values are reached very quickly. Next, the number of chosen cities stabilizes (Fig. 12).

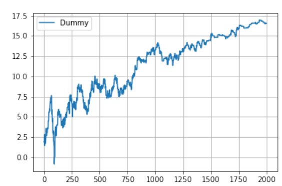

Fig. 5: Objective function

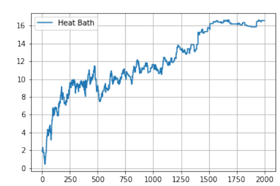

Fig. 6: Objective function

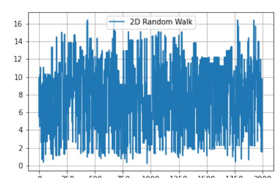

Fig. 7: Objective function

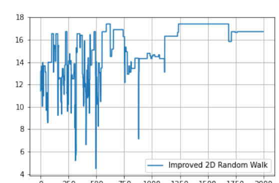

Fig. 8: Objective function

During the optimization, the best state is the one with the highest value of the objective function. Figures 13 - 16 exhibit the best states for all base chains. In the figures, the cities are represented based on their x coordinates. The size of the point (city) represents the population of a city. In the best states, Dummy chain and Heat-Bath chain tend to cover a large surface of the country (Fig. 13 and Fig 14), since they, by design, do not include all cities in the covered circle. They act greedily by trying to reach the most populated cities. However, even maximal function values are not much different, two other chains completely cover circle areas that are smaller (Fig. 15 and Fig. 16).

C. The cardinality of S^* and the function value

Figures 17 and 18 are obtained with the Heat Bath base chain. We choose only one chain for this part of the task since it is computationally intensive and, concretely, the Heat Bath

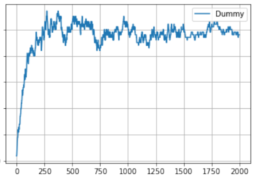

Fig. 9: No. of cities

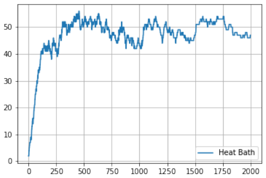

Fig. 10: No. of cities

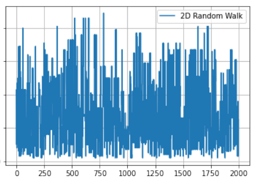

Fig. 11: No. of cities

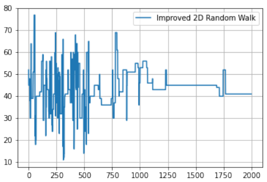

Fig. 12: No. of cities

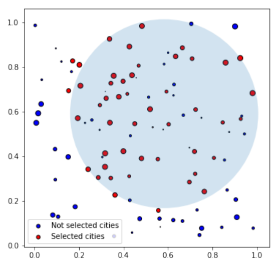

Fig. 13: Best state for Dummy chain

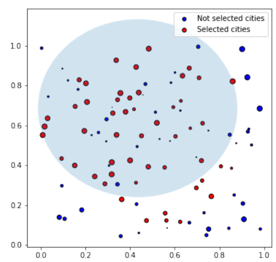

Fig. 14: Best state for Heat-bath chain

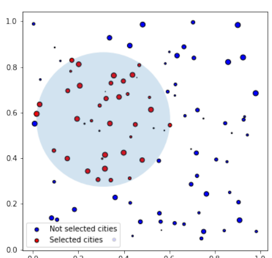

Fig. 15: Best state for 2D RW chain

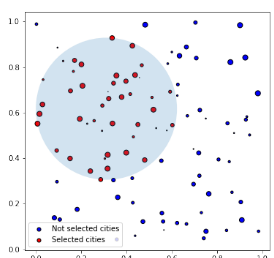

Fig. 16: Best state for Imp. 2D RW chain

base chain exhibits convergence for both the number of cities and values of the objective function.

We compute the empirical means for 10 different λ in the interval between 0 and 1. The number of datasets is 15 and the chain evolves during 600 steps. Parameters are set according the previous experiments and differ for distinct λ.

In general, these plots decrease in λ. This is sensible because the higher λ directly diminishes the objective function's value and does not allow choosing many cities. However, one may notice that the empirical mean value of the objective function is not linear. This slowdown in λ is related to the underlying ideas of the base chain. The heat bath base chain

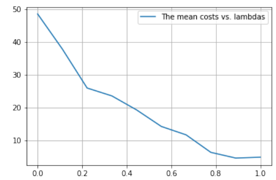

Fig. 17: The obejctive function mean vs. λ

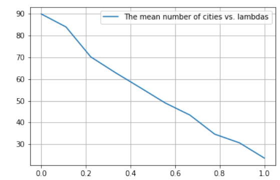

Fig. 18: No. cities mean vs. λ

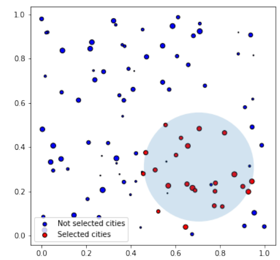

Fig. 19: The best state for $\lambda = 0.9$

favors states with larger values of the objective function. Thus, it focuses on the highest populated cities. In the case of uniform distribution, it jumps between areas of cities with a big population. Therefore, when we start from zero and slightly increase λ, the chain strives to discard all cities with a small population. Further, removing all such cities makes it more challenging for the chain to remove cities with big populations because it has a bigger weight. Thus, approaching to 1 in λ, it keeps several of the biggest cities that can not be excluded. Otherwise, the chain leaves the optimal state. It can be seen in Fig. 19 and 14. In the first plot, the state does not contain small-populated cities, whereas the second one does it.

We see that Fig. 17 has three main slopes. The first one is the interval between 0.8 and 1.0. There, the objective function does not depend on a value of λ. Thus, we can put $\lambda = 0.8$ and choose more cities because it allows us to collect more feedback and does not affect the value of the objective function. Then, the next interval is $[0.2, 0.8]$. Here, we can use steps that are proportional to the slope. In this way, we can quickly change our strategy if we get bad feedback from consumers. In the third slope $[0.0, 0.2]$, we propose that λ should change with big steps since the objective function changes more radically. At this point, feedback from customers is good, the practical part is almost tuned, and the company can afford to extend the deploying area sharply.

IV. Analysis on G2

A. *Choosing the best base chain and tuning parameters*

Plots 20 and 21 show the dependency of objective function and number of chosen cities on λ parameter correspondingly. β and δ were chosen for each chain and each λ according to the best objective function value. According to Fig. 20, we conclude that the Heat Bath base chain outperforms other ones, except 2D RW. Now considering Fig. 21 one can notice more stable behavior of the Heat Bath base chain than 2D RW one. As a result, for this dataset Heat Bath chain is a reasonable choice too.

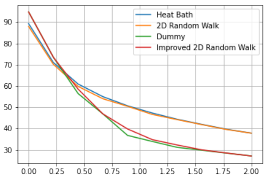

Fig. 20: Function vs. λ

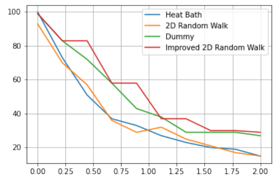

Fig. 21: No. of cities vs. λ

B. Algorithm behavior through steps

Figures 22-29 depict results for $\lambda = 1.11$. Parameters for simulated annealing are tuned for each base chain.

Similarly, as with dataset G1, the function value steadily increases and saturates for Dummy chain (Fig. 22). The number of chosen cities reach 40 quickly, and then it oscillates over time (Fig. 26). With the Heat-Bath the function value is the highest (Fig. 23). Even though the Heat-Bath chain initially explores a bad region, it finds better function values when it starts to decrease the number of chosen cities, around step 250 (Fig. 27).

For two other chains, the best function values are around 35. 2D RW again does not have stable behaviour (Fig. 24 and Fig. 28). Similarly, with its improved version, the behavior is improved (Fig. 25 and Fig. 29).

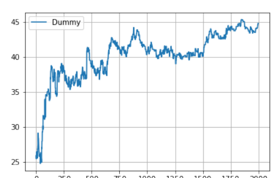

Fig. 22: Objective function

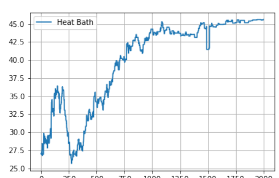

Fig. 23: Objective function

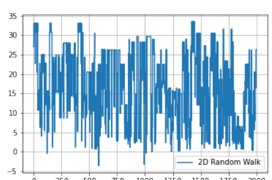

Fig. 24: Objective function

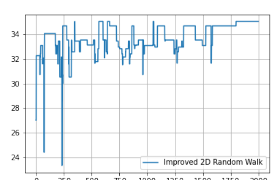

Fig. 25: Objective function

Figures 30 and 31 exhibit best states for dataset G2. Recall that the initial state is chosen such that the two most populated cities are included in the solution. It can be seen that these cities are included in the best state. For the distribution G2, the dummy chain also greedily tries to find populated cities not concerning the possible change in the maximal diameter.

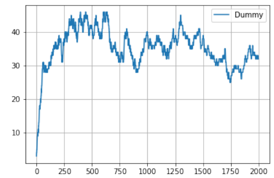

Fig. 26: Number of cities

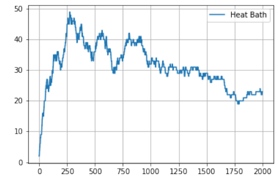

Fig. 27: Number of cities

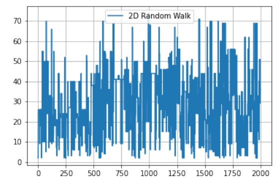

Fig. 28: Number of cities

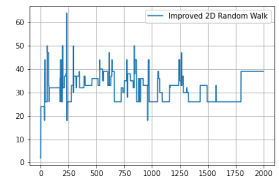

Fig. 29: Number of cities

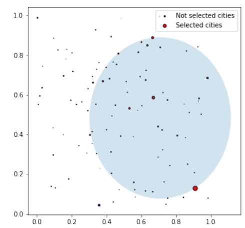

Fig. 30: Best state for Dummy chain

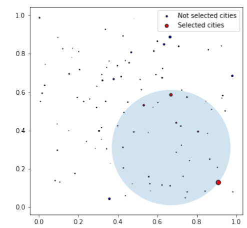

Fig. 31: Best state for 2D RW chain

C. The cardinality of S^* and the function value

The figures 32 and 33 exhibit the empirical means in the case of Gaussian distribution. Experiment parameters, in this case, are the same as in experiments with uniform distribution.

In this case, fewer cities have big populations. Hence, there are fewer local optimal points. Thus, there is a slowdown in Fig.33. The algorithm is forced to properly selects with the growth of λ. However, a choice of cities is not various and the optimal area is well-defined. Thus, the chain usually finds these cities and, depending on λ adds more or less smaller cities. The jumps in Fig.32 therefore can be explained in the following way. It may be caused only by either population of chosen cities or their locations because Fig. 33 does not have such jumps. Specifically, the transition from $\lambda = 0.66$ (Fig.34) to $\lambda = 0.88$ (Fig.35) results in discarding one small far city that was chosen because of the value of λ. This discarding diminishes the area and proposes a better state. This is less likely in the case of uniform distribution since there we more likely substitute one city with another one.

In the case of dataset G2, the company should have uniform steps at decreasing λ, meaning that it should equally wait to gather the feedback and then enlarge the surface area it covers. Here, we do not have many extremely-populated cities. The initial state already chooses the two most populated cities. Therefore, expanding to new areas should be done carefully.

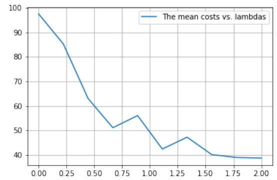

Fig. 32: The objective function mean vs. λ

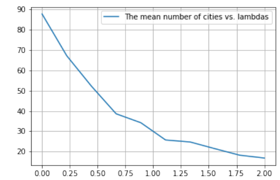

Fig. 33: No. cities mean vs. λ

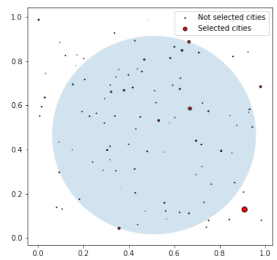

Fig. 34: The best state for $\lambda = 0.66$

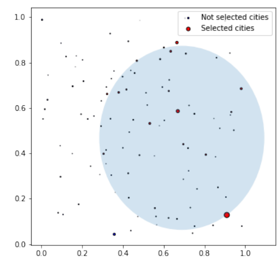

Fig. 35: The best state for $\lambda = 0.88$

V. Conclusion

Comparing four proposed base chains results in us choosing the chain based on the heat bath dynamics. It showed one of the best values for the objective function and achieved convergence in both the number of cities and the value of the objective function. We obtained the plots of empirical means and could build the schedule of λ with this chain.

However, Heat-Bath chain has an exponential number of states in terms of cities. We proposed Improved 2D Random Walk that has dramatically reduced the number of states and performs the walk more quickly. It showed competitive results in function optimization and it has more robust behavior than 2D Random Walk.