Integral: Techniques Integration

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- **Rasional Reduction (improper fraction)**
- Improper fraction: the degree of numerator is greater than or equal to the degree of denominator
- example:

$$\int \frac{x^2+2x+2}{x^2+2}\,dx$$

$$\begin{array}{r} 1 \\ x^2+2\overline{\big)\, x^2+2x+2} \\ \underline{x^2+2} \\ 2x \end{array}$$

$$\frac{x^2+2x+2}{x^2+2} = 1 + \frac{2x}{x^2+2}$$

Exercises:

1. $\displaystyle \int \frac{2x^3}{x^2-1}\,dx$

2. $\displaystyle \int \frac{4t^3-t^2+16t}{t^2+4}\,dt$ 3. $\displaystyle \int \frac{4x^2-7}{2x+3}\,dx$

Rasional Separation

$$\int \frac{x+2}{x^2+1}dx = \int \frac{x}{x^2+1}dx + \int \frac{2}{x^2+1}dx$$

$Exercises:$

1. $\displaystyle\int \frac{1+\sin x}{\cos^2 x}\, dx$
2. $\displaystyle\int \frac{1-x}{\sqrt{1-x^2}}\, dx$
3. $\displaystyle\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\, dx$

Multiplication by 1

$$\int \sec x \, dx$$

$$solution \quad :$$

$$\int \sec x \, dx = \int \sec x . \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$example: \quad u = \sec x + \tan x$$

$$du = (\sec^2 x + \sec x \tan x) dx$$

$$\int \sec x \, dx = \int \frac{du}{u} = \ln |u| + C$$

$$= \ln |\sec x + \tan x| + C$$

Exercises

1. $\displaystyle\int \frac{1}{1+\sin x}\,dx$

2. $\displaystyle\int \frac{1}{\csc x+\cot x}\,dx$

3. $\displaystyle\int \frac{1}{\sec x+\tan x}\,dx$

Integration by part

- Integral of multiplication of two functions is not the same with the multiplication of two integrals of those two functions:

- example: $\int f(x).g(x)dx \neq \int f(x)dx.\int g(x)dx$

$$\int x^2 dx \neq \int x\, dx.\int x\, dx$$

Integration by part

- From derivation rules:

$$\frac{d}{dx}\left[f(x).g(x)\right] = \frac{df(x)}{dx}.g(x) + f(x).\frac{dg(x)}{dx}$$

$$or:$$

$$\frac{d}{dx}\left[f(x).g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

$$\int \frac{d}{dx}\left[f(x).g(x)\right]dx = \int \left[f'(x)g(x) + f(x)g'(x)\right]dx$$

$$f(x).g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Integration by part

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du$$

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx$$

Example: integration by part

$$the\ choice\ of\ u\ and\ dv$$

$$1.\ u = 1; dv = x\cos x\, dx$$

$$2.\ u = x; dv = \cos x\, dx$$

$$3.\ u = x\cos x; dv = dx$$

$$4.\ u = \cos x; dv = x\, dx$$

$$\int x\cos x dx$$

$$example: u = x \qquad dv = \cos x\, dx$$

$$du = dx \qquad v = \sin x$$

$$\int x\cos x dx = x\sin x - \int \sin x\, dx = x\sin x + \cos x + C$$

exercises

1. $\int \ln x \, dx$
2. $\int x^2 e^x dx$
3. $\int e^x \cos x \, dx$
4. $\int x \ln x \, dx$

Integral by part: tabulation

$$\int x^3 \sin x \, dx, \quad f(x) = x^3, \quad g(x) = \sin x$$

***f*(*x*) and its derivatives**		***g*(*x*) and its integrals**
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

example

$$\int x^3 \sin x \, dx$$

- f(x) =x^3 and g(x)=sin x

f(x) and derivation		g(x) and integral
x^3	+	sin x
$3x^2$	-	- cos x
6x	+	- sin x
6	-	cos x
0		sin x

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

Trigonometri

- Multiplication of sinus and cosinus

$$\int \sin^m x \cos^n x \, dx$$

- where m and n are non negative integers

1. If m is odd, then m=2k+1 and identity $\sin^2 x = 1 - \cos^2 x$

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x$$

2. If m is even and n is odd, n = 2k + 1 and identity

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x$$

Integral Trigonometri

3. If both are even (m and n), we substitute

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad dan \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

examples

- if m is odd

$$\int \sin^3 x \cos^2 x\, dx = \int \sin^2 x \cos^2 x \sin x\, dx = \int (1-\cos^2 x)\cos^2 x \sin x\, dx$$

$$let\ u = \cos x, \quad then\ du = -\sin x dx$$

$$= -\int (1-u^2)u^2 du = -\int (u^2 - u^4) du = -\frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

- if m is even and n is odd

$$\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1-\sin^2 x)^2 \cos x\, dx$$

$$misal \quad u = \sin x$$

$$= \int (1-u^2)^2 du = \int (1-2u^2+u^4) du$$

$$= u - \frac{2}{3}u^3 + \frac{u^5}{5} + C = \sin x - \frac{2}{3}\sin^3 x + \frac{\sin^5 x}{5} + C$$

example

- if m and n are even

$$\int \sin^4 x \cos^2 x \, dx = \int (\sin^2 x)^2 \cos^2 x \, dx$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{8} \int (1 - 2\cos 2x + \cos^2 2x)(1 + \cos 2x) \, dx$$

$$= \frac{1}{8} \int \left(1 + \cos 2x - 2\cos 2x + \cos^2 2x - 4\cos^2 2x + \cos^3 2x\right) dx$$

$$= \frac{1}{8} [x - \frac{1}{2}\sin 2x + \int \left(\cos^2 2x - 4\cos^3 2x\right) dx]$$

$Part 1:$

$$\int \cos^2 2x \, dx = \frac{1}{2} \int (1 + \cos 4x) \, dx = \frac{1}{2}(x + \frac{1}{4}\sin 4x)$$

$part 2:$

$$-4 \int \cos^3 2x \, dx = \int (1 - \sin^2 2x) \cos 2x \, dx$$

$let \; u = \sin 2x$

$$-4 \int \cos^3 2x \, dx = -2 \int (1 - u^2) du = -2\left(x - \frac{1}{3}\sin^3 2x\right)$$

$so:$

$$\int \sin^4 x \cos^2 x \, dx = \frac{1}{8}\left[x - \frac{1}{2}\sin 2x + \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) - 2\left(x - \frac{1}{3}\sin^3 2x\right)\right] + C$$

Multiplication of tan x and sec x

- example: $\int \tan^4 x\,dx = \int \tan^2 x.\tan^2 x\,dx = \int \tan^2 x(\sec^2 x - 1)\,dx$

$$= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$$

$$= \int \tan^2 x \sec^2 x\,dx - \int (\sec^2 x - 1)\,dx$$

for

$$\int \tan^2 x \sec^2 x\,dx$$

$$let\ u = \tan x, \quad du = \sec^2 x\,dx$$

$$\int \tan^2 x \sec^2 x\,dx = \int u^2 du = \frac{1}{3}u^3 + C_1$$

so

$$\int \tan^4 x\,dx = \frac{1}{3}\tan x - \tan x + x + C$$

exercises

1. $\int \cos^3 x \, dx$

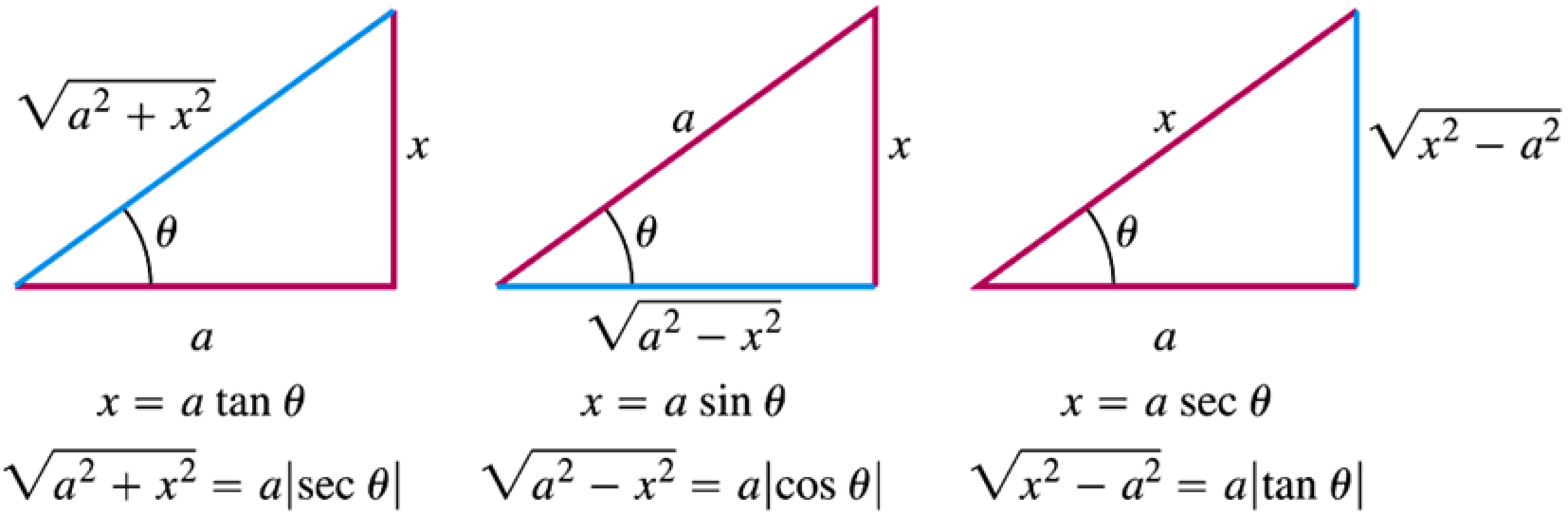
2. $\int \sin^4 x \, dx$

3. $\int 2\sec^3 x \, dx$

4. $\int 4\tan^3 x \, dx$

5. $\int \sin^2 2\theta \cos^3 2\theta \, d\theta$

Trigonometri substitution



$$if\ \frac{1}{\sqrt{a^2+x^2}};\ use\quad x = a\tan\theta$$

$$if\ \frac{1}{\sqrt{a^2-x^2}};\ use\quad x = a\sin\theta$$

$$if\ \frac{1}{\sqrt{x^2-a^2}};\ use\quad x = a\sec\theta$$

exercises

1. $$\int \frac{dx}{\sqrt{4+x^2}}$$

2. $$\int \frac{dx}{\sqrt{9-x^2}}$$

3. $$\int \frac{dx}{\sqrt{9x^2-4}}$$

4. $$\int \sqrt{1-9t^2}\; dt$$

5. $$\int \frac{\sqrt{9-w^2}}{w^2}\; dw$$

6. $$\int \frac{x\, dx}{\sqrt{x^2-1}}$$

7. $$\int \frac{dr}{r^2\sqrt{r^2+1}}$$

Rational Functions

- If integral function is rational, then check whether the function is proper or improper.
- if improper, then do division.
- If proper, then do partial fraction

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

examples

- Different Linear Factor

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

- Repeated Linear factor

$$\int \frac{6x+7}{(x+2)^2}\,dx$$

$$\frac{6x+7}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

examples

- Improper fraction

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\, dx$$

- Quadratic factor

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)}\, dx$$

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1}$$

Heaviside Method

Heaviside Method

1. *Write the quotient with $g(x)$ factored:*

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x - r_1)(x - r_2)\cdots(x - r_n)}.$$

2. *Cover the factors $(x - r_i)$ of $g(x)$ one at a time*, each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:

$$A_1 = \frac{f(r_1)}{(r_1 - r_2)\cdots(r_1 - r_n)}$$

$$A_2 = \frac{f(r_2)}{(r_2 - r_1)(r_2 - r_3)\cdots(r_2 - r_n)}$$

$$\vdots$$

$$A_n = \frac{f(r_n)}{(r_n - r_1)(r_n - r_2)\cdots(r_n - r_{n-1})}.$$

3. *Write the partial-fraction expansion of $f(x)/g(x)$* as

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

example

- With heaviside

$$\int \frac{x+4}{x^3+3x^2-10x}\,dx$$

$$\frac{x+4}{x^3+3x^2-10x} = \frac{x+4}{x(x-2)(x+5)}$$

$$r_1 = 0;\ r_2 = 2;\ r_3 = -5$$

$$A_1 = \frac{f(r_1)}{(r_1-r_2)(r_1-r_3)}$$

$$A_2 = \frac{f(r_2)}{(r_2-r_1)(r_2-r_3)}$$

$$A_3 = \frac{f(r_3)}{(r_3-r_1)(r_3-r_2)}$$