# 第一章　绪论

## 1.1　定义

## 1.2　课程内容

## 1.3　教学安排

# 第二章　控制方程

## 2.1　纳维-斯托克斯方程

### 2.1.1　模型

### 2.1.2　连续性方程

### 2.1.3　运动方程

## 2.2　雷诺应力平均方程

### 2.2.1　紊流基础特征

### 2.2.2　紊流连续性方程

### 2.2.3　紊流运动方程

### 2.2.4　紊流模型

## 2.3　平面二维浅水方程

天然河道水流运动一般都属于三维流动，运动要素即沿程变化，又沿水深和河宽方向变化。由于三维水流运动比较复杂，河流数值模拟常用的一种简化方法是将运动要素沿水深方向平均，把三维问题转化为平面二维问题。本节基于一定条件将三维流动的雷诺平均运动微分方程简化为平面二维浅水方程。

图 2.1: 水位基准示意图

### 2.3.1 浅水假设和水深平均积分法则

浅水假设

三维流动的雷诺平均运动微分方程如式(2.1a)-(2.1d)所示。

$$\frac{\partial \overline{u_x}}{\partial x}+\frac{\partial \overline{u_y}}{\partial y}+\frac{\partial \overline{u_z}}{\partial z}=0 \tag{2.1a}$$

$$\frac{\partial \overline{u_x}}{\partial t}+\frac{\partial (\overline{u_x}\overline{u_x})}{\partial x}+\frac{\partial (\overline{u_x}\overline{u_y})}{\partial y}+\frac{\partial (\overline{u_x}\overline{u_z})}{\partial z}=\bar{f}_x-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\nu_t\left(\frac{\partial^2 \overline{u_x}}{\partial x^2}+\frac{\partial^2 \overline{u_x}}{\partial y^2}+\frac{\partial^2 \overline{u_x}}{\partial z^2}\right) \tag{2.1b}$$

$$\frac{\partial \overline{u_y}}{\partial t}+\frac{\partial (\overline{u_y}\overline{u_x})}{\partial x}+\frac{\partial (\overline{u_y}\overline{u_y})}{\partial y}+\frac{\partial (\overline{u_y}\overline{u_z})}{\partial z}=\bar{f}_y-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y}+\nu_t\left(\frac{\partial^2 \overline{u_y}}{\partial x^2}+\frac{\partial^2 \overline{u_y}}{\partial y^2}+\frac{\partial^2 \overline{u_y}}{\partial z^2}\right) \tag{2.1c}$$

$$\frac{\partial \overline{u_z}}{\partial t}+\frac{\partial (\overline{u_z}\overline{u_x})}{\partial x}+\frac{\partial (\overline{u_z}\overline{u_y})}{\partial y}+\frac{\partial (\overline{u_z}\overline{u_z})}{\partial z}=\bar{f}_z-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z}+\nu_t\left(\frac{\partial^2 \overline{u_z}}{\partial x^2}+\frac{\partial^2 \overline{u_z}}{\partial y^2}+\frac{\partial^2 \overline{u_z}}{\partial z^2}\right) \tag{2.1d}$$

在河道、湖泊或水库水流中，水平尺度一般远大于垂向尺度。如果垂向加速度与重力加速度相比很小，则可以忽略垂向加速度，流速等水力参数沿垂向的变化常采用其垂向平均值，并假定沿水深方向的动水压强分布符合静水压强分布。三维流动的运动微分方程可简化为：

$$\frac{\partial \overline{u_x}}{\partial x}+\frac{\partial \overline{u_y}}{\partial y}+\frac{\partial \overline{u_z}}{\partial z}=0 \tag{2.2a}$$

$$\frac{\partial \overline{u_x}}{\partial t}+\frac{\partial (\overline{u_x}\overline{u_x})}{\partial x}+\frac{\partial (\overline{u_x}\overline{u_y})}{\partial y}+\frac{\partial (\overline{u_x}\overline{u_z})}{\partial z}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\nu_t\left(\frac{\partial^2 \overline{u_x}}{\partial x^2}+\frac{\partial^2 \overline{u_x}}{\partial y^2}+\frac{\partial^2 \overline{u_x}}{\partial z^2}\right) \tag{2.2b}$$

$$\frac{\partial \overline{u_y}}{\partial t}+\frac{\partial (\overline{u_y}\overline{u_x})}{\partial x}+\frac{\partial (\overline{u_y}\overline{u_y})}{\partial y}+\frac{\partial (\overline{u_y}\overline{u_z})}{\partial z}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y}+\nu_t\left(\frac{\partial^2 \overline{u_y}}{\partial x^2}+\frac{\partial^2 \overline{u_y}}{\partial y^2}+\frac{\partial^2 \overline{u_y}}{\partial z^2}\right) \tag{2.2c}$$

$$\frac{\partial \bar{p}}{\partial z}=-\rho g \tag{2.2d}$$

水深积分平均法则

将式(2.2)沿水深积分平均，即可得到沿水深平均的平面二维流动的基本方程。在沿水深积分平均过程中，采用以下定义和公式：

（1）定义水深为

$$H=\zeta-z_0 \tag{2.3}$$

式中，$H$ 为水深，$\zeta=\zeta(x,y,t)$、$z_0=z_0(x,y,t)$ 分别为某一基准面下的水面高程和河床高程（见图）

（2）定义沿水深平均流速 $U_i$ 为

$$U_i=\frac{1}{H}\int_{z_0}^{\zeta}\overline{u_i}\mathrm{d}z \tag{2.4}$$

式中，下标 $i$ 取 1，2 和 3 分别对应 $x$，$y$ 和 $z$ 方向的速度分量。

（3）莱布尼兹公式

$$\frac{\partial}{\partial x_i}\int_a^b f\mathrm{d}z=\int_a^b\frac{\partial f}{\partial x_i}\mathrm{d}z+f|_b\frac{\partial b}{\partial x_i}-f|_a\frac{\partial a}{\partial x_i} \tag{2.5}$$

式中，$a$、$b$ 和 $f$ 都是 $x_i$ 的函数。

（4）自由表面及河床底部运动学条件为：

$$\overline{u_z}|_{z=\zeta} = \frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + \overline{u_x}|_{z=\zeta}\frac{\partial \zeta}{\partial x} + \overline{u_y}|_{z=\zeta}\frac{\partial \zeta}{\partial y} \tag{2.6}$$

$$\overline{u_z}|_{z=z_0} = \frac{Dz_0}{Dt} = \frac{\partial z_0}{\partial t} + \overline{u_x}|_{z=z_0}\frac{\partial z_0}{\partial x} + \overline{u_y}|_{z=z_0}\frac{\partial z_0}{\partial y} \tag{2.7}$$

### 2.3.2 沿水深平均的连续性方程

采用上述定义和公式对连续性方程(2.2a)沿水深积分平均得：

$$\int_{z_0}^{\zeta}\left(\frac{\partial \overline{u_x}}{\partial x} + \frac{\partial \overline{u_y}}{\partial y} + \frac{\partial \overline{u_z}}{\partial z}\right)\mathrm{d}z = 0 \tag{2.8}$$

根据式(2.5)，式(2.8)中前两项分别可写成

$$\int_{z_0}^{\zeta}\frac{\partial \overline{u_x}}{\partial x}\mathrm{d}z = \frac{\partial}{\partial x}\int_{z_0}^{\zeta}\overline{u_x}\mathrm{d}z - \overline{u_x}|_{z=\zeta}\frac{\partial \zeta}{\partial z} + \overline{u_x}|_{z=z_0}\frac{\partial z_0}{\partial z} \tag{2.9}$$

$$\int_{z_0}^{\zeta}\frac{\partial \overline{u_y}}{\partial y}\mathrm{d}z = \frac{\partial}{\partial y}\int_{z_0}^{\zeta}\overline{u_y}\mathrm{d}z - \overline{u_y}|_{z=\zeta}\frac{\partial \zeta}{\partial z} + \overline{u_y}|_{z=z_0}\frac{\partial z_0}{\partial z} \tag{2.10}$$

式(2.8)中最后一项

$$\int_{z_0}^{\zeta}\frac{\partial \overline{u_z}}{\partial z}\mathrm{d}z = \overline{u_z}|_{z=\zeta} - \overline{u_z}|_{z=z_0} \tag{2.11}$$

将式(2.9)，(2.10)和(2.11) 代入(2.8)，并利用自由表面及河床底部运动学条件式(2.6) 和(2.7)，可得

$$\begin{aligned}
&\int_{z_0}^{\zeta}\left(\frac{\partial \overline{u_x}}{\partial x} + \frac{\partial \overline{u_y}}{\partial y} + \frac{\partial \overline{u_z}}{\partial z}\right)\mathrm{d}z \\
=&\frac{\partial}{\partial x}\int_{z_0}^{\zeta}\overline{u_x}\mathrm{d}z - \overline{u_x}|_{z=\zeta}\frac{\partial \zeta}{\partial z} + \overline{u_x}|_{z=z_0}\frac{\partial z_0}{\partial z}+ \\
&\frac{\partial}{\partial y}\int_{z_0}^{\zeta}\overline{u_y}\mathrm{d}z - \overline{u_y}|_{z=\zeta}\frac{\partial \zeta}{\partial z} + \overline{u_y}|_{z=z_0}\frac{\partial z_0}{\partial z}+ \\
&\overline{u_z}|_{z=\zeta} - \overline{u_z}|_{z=z_0} \\
=&\frac{\partial HU_x}{\partial x} + \frac{\partial HU_y}{\partial x} + \frac{\partial \zeta}{\partial t} - \frac{\partial z_0}{\partial t} = 0
\end{aligned}$$

最后得

$$\frac{\partial H}{\partial t} + \frac{\partial HU_x}{\partial x} + \frac{\partial HU_y}{\partial y} = 0 \tag{2.12}$$

也可以写成

$$\frac{\partial H}{\partial t} + \nabla\cdot(H\mathbf{U}) = 0 \tag{2.13}$$

### 2.3.3 沿水深平均的运动方程

以 $x$ 方向为例，式(2.2b)沿水深积分为

$$\int_{z_0}^{\zeta}\left[\frac{\partial \overline{u_x}}{\partial t} + \frac{\partial(\overline{u_x}\,\overline{u_x})}{\partial x} + \frac{\partial(\overline{u_x}\,\overline{u_y})}{\partial y} + \frac{\partial(\overline{u_x}\,\overline{u_z})}{\partial z} + \frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} - \nu_t\left(\frac{\partial^2 \overline{u_x}}{\partial x^2} + \frac{\partial^2 \overline{u_x}}{\partial y^2} + \frac{\partial^2 \overline{u_x}}{\partial z^2}\right)\right]\mathrm{d}z = 0 \tag{2.14}$$

式(2.14)中包含了非恒定流项积分、对流项积分、压力项积分和阻力项积分。接下来分项讨论。

（1）非恒定流项积分

$$\begin{aligned}&\int_{z_0}^{\zeta}\frac{\partial\overline{u_x}}{\partial t}\mathrm{d}z=\frac{\partial}{\partial t}\int_{z_0}^{\zeta}\overline{u_x}\mathrm{d}z-\overline{u_x}|_{z=\zeta}\frac{\partial\zeta}{\partial t}+\overline{u_x}|_{z=z_0}\frac{\partial z_0}{\partial t}\\=&\frac{\partial HU_x}{\partial t}-\overline{u_x}|_{z=\zeta}\frac{\partial\zeta}{\partial t}+\overline{u_x}|_{z=z_0}\frac{\partial z_0}{\partial t}\end{aligned}\tag{2.15}$$

（2）对流项积分

首先将时均流速按照式(2.16)进行分解

$$\begin{aligned}\overline{u_x}&=U_x+\Delta\overline{u_x}\\\overline{u_y}&=U_y+\Delta\overline{u_y}\end{aligned}\tag{2.16}$$

式中，$\Delta\overline{u_x}$ 和 $\Delta\overline{u_y}$ 分别为 $x$ 和 $y$ 的时均流速与垂线平均流速的差值。

对流项中第一项的积分

$$\int_{z_0}^{\zeta}\frac{\partial\overline{u_x}\overline{u_x}}{\partial x}\mathrm{d}z=\frac{\partial}{\partial x}\int_{z_0}^{\zeta}\overline{u_x}\overline{u_x}\mathrm{d}z-\overline{u_x}\overline{u_x}|_{z=\zeta}\frac{\partial\zeta}{\partial x}+\overline{u_x}\overline{u_x}|_{z=z_0}\frac{\partial z_0}{\partial x}\tag{2.17}$$

式中

$$\begin{aligned}\int_{z_0}^{\zeta}\overline{u_x}\overline{u_x}\mathrm{d}z&=\int_{z_0}^{\zeta}(U_x+\Delta\overline{u_x})(U_x+\Delta\overline{u_x})\mathrm{d}z\\&=\int_{z_0}^{\zeta}(U_xU_x+2U_x\Delta\overline{u_x}+\Delta\overline{u_x}\Delta\overline{u_x})\mathrm{d}z\\&=HU_xU_x+\int_{z_0}^{\zeta}\Delta\overline{u_x}\Delta\overline{u_x}\mathrm{d}z\\&=\beta_{xx}HU_xU_x\end{aligned}$$

其中

$$\beta_{xx}=1+\frac{\int_{z_0}^{\zeta}\Delta\overline{u_x}\Delta\overline{u_x}\mathrm{d}z}{HU_xU_x}\tag{2.18}$$

是由于流速沿垂线分布不均匀而引入的修正系数，类似于水力学中的动量修正系数。$\beta_{xx}$ 的取值一般在 1.02 与 1.05 之间，可近似取为 1.0。因此

$$\int_{z_0}^{\zeta}\frac{\partial\overline{u_x}\overline{u_x}}{\partial x}\mathrm{d}z=\frac{\partial HU_xU_x}{\partial x}-\overline{u_x}\overline{u_x}|_{z=\zeta}\frac{\partial\zeta}{\partial x}+\overline{u_x}\overline{u_x}|_{z=z_0}\frac{\partial z_0}{\partial x}\tag{2.19}$$

同理，对流项的第二项可写为

$$\int_{z_0}^{\zeta}\frac{\partial\overline{u_x}\overline{u_y}}{\partial y}\mathrm{d}z=\frac{\partial HU_xU_y}{\partial x}-\overline{u_x}\overline{u_y}|_{z=\zeta}\frac{\partial\zeta}{\partial y}+\overline{u_x}\overline{u_y}|_{z=z_0}\frac{\partial z_0}{\partial y}\tag{2.20}$$

对流项的第三项

$$\int_{z_0}^{\zeta}\frac{\partial\overline{u_x}\overline{u_z}}{\partial z}\mathrm{d}z=\overline{u_x}\overline{u_z}|_{z=\zeta}-\overline{u_x}\overline{u_z}|_{z=z_0}\tag{2.21}$$

将非恒定流项和对流项积分相加，并利用自由表面和河床底部运动学条件可得：

$$\int_{z_0}^{\zeta}\left[\frac{\partial\overline{u_x}}{\partial t}+\frac{\partial(\overline{u_x}\overline{u_x})}{\partial x}+\frac{\partial(\overline{u_x}\overline{u_y})}{\partial y}+\frac{\partial(\overline{u_x}\overline{u_z})}{\partial z}\right]\mathrm{d}z=\frac{\partial HU_x}{\partial t}+\frac{\partial HU_xU_x}{\partial x}+\frac{\partial HU_xU_y}{\partial y}\tag{2.22}$$

（3）压力项积分

$$\begin{aligned}\int_{z_0}^{\zeta}\frac{\partial\overline{p}}{\partial x}\mathrm{d}z&=\frac{\partial}{\partial x}\int_{z_0}^{\zeta}\overline{p}\mathrm{d}z-\overline{p}|_{z=\zeta}\frac{\partial\zeta}{\partial x}+\overline{p}|_{z=z_0}\frac{\partial z_0}{\partial x}\\&=\frac{\partial}{\partial x}\int_{z_0}^{\zeta}\rho g(\zeta-z)\mathrm{d}z-\rho g(\zeta-z)|_{z=\zeta}\frac{\partial\zeta}{\partial x}+\rho g(\zeta-z)|_{z=z_0}\frac{\partial z_0}{\partial x}\\&=\rho gH\frac{\partial H}{\partial x}+\rho gH\frac{\partial z_0}{\partial x}=\rho gH\frac{\partial\zeta}{\partial x}\end{aligned}\tag{2.23}$$

（4）阻力项积分

### 2.3.4 浅水方程形式

## 2.4 一维圣维南方程

### 2.4.1 一维连续性方程

### 2.4.2 一维运动方程

### 2.4.3 圣维南方程形式

# 第三章　有限差分法

## 3.1　泰勒展开

## 3.2　离散基础知识

## 3.3　差分方程

## 3.4　相容性、稳定性和收敛性

# 第四章　有限体积法

## 4.1　一维热传导方程的有限体积法

## 4.2　一维对热扩散方程的有限体积法

# 第五章　一维水动力学问题

## 5.1　一维恒定流求解

## 5.2　一维非恒定流求解

# 第六章　二维水动力学问题

## 6.1　网格基础

## 6.2　二维浅水方程求解