

第一章 绪论

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2.2.1 紊流物理量时均值定义及性质

根据大量相关实验观测，紊流具有有涡性、不规则性、耗能性、连续性、三维性以及非定常性等特征。因此，紊流运动要素都具有随机性。为了后续分析方便，紊流运动要素的瞬时值常被分解成时均值和脉动值。任一运动要素 ϕ 的时均值定义为：

$$\bar{\phi} = \frac{1}{\Delta t} \int_t^{t+\Delta t} \phi(t) dt \quad (2.1)$$

其中，时间间隔 Δt 相对于紊流的随机脉动周期而言足够大，但相对于流场的各种时均量的缓慢变化周期来说，应足够小。如果时均值随时间变化，成为非稳态的时均紊流；如果时均值不随时间变化，成为准稳态紊流，简称稳态紊流。

运动要素的瞬时值 ϕ ，时均值 $\bar{\phi}$ 及脉动值 ϕ' 之间有如下关系：

$$\phi = \bar{\phi} + \phi' \quad (2.2)$$

设 ϕ 和 f 是两个瞬时值， $\bar{\phi}$ 和 \bar{f} 为相应的时均值， ϕ' 和 f' 为相应的脉动值。按照式(2.1)和(2.2)，下列基本关系成立：

$$\begin{aligned} \bar{\phi'} &= 0 & \bar{\bar{\phi}} &= \bar{\phi} & \overline{\bar{\phi} + \phi'} &= \bar{\phi} \\ \overline{\phi f} &= \bar{\phi} \bar{f} & \overline{\phi \bar{f}} &= \bar{\phi} \bar{f} & \overline{\phi f'} &= 0 & \overline{\phi f} &= \bar{\phi} \bar{f} + \overline{\phi' f'} \\ \frac{\partial \bar{\phi}}{\partial x_i} &= \frac{\partial \bar{\phi}}{\partial x_i} & \frac{\partial^2 \bar{\phi}}{\partial x_i^2} &= \frac{\partial^2 \bar{\phi}}{\partial x_i^2} & \frac{\partial \bar{\phi'}}{\partial x_i} &= 0 & \frac{\partial^2 \bar{\phi'}}{\partial x_i^2} &= 0 \end{aligned} \quad (2.3)$$

2.2.2 雷诺时均连续性方程

2.2.3 雷诺时均运动方程

2.2.4 紊流模型

2.3 平面二维浅水方程

天然河道水流运动一般都属于三维流动，运动要素即沿程变化，又沿水深和河宽方向变化。由于三维水流运动比较复杂，河流数值模拟常用的一种简化方法是将运动要素沿水深方向平均，把三维问题转化为平面二维问题。本节基于一定条件将三维流动的雷诺平均运动微分方程简化为平面二维浅水方程。

2.3.1 浅水假设和水深平均积分法则

浅水假设

三维流动的雷诺平均运动微分方程如式(2.4a)-(2.4d)所示。

$$\frac{\partial \bar{u}_x}{\partial x} + \frac{\partial \bar{u}_y}{\partial y} + \frac{\partial \bar{u}_z}{\partial z} = 0 \quad (2.4a)$$

$$\frac{\partial \bar{u}_x}{\partial t} + \frac{\partial(\bar{u}_x \bar{u}_x)}{\partial x} + \frac{\partial(\bar{u}_x \bar{u}_y)}{\partial y} + \frac{\partial(\bar{u}_x \bar{u}_z)}{\partial z} = \bar{f}_x - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + \nu_t \left(\frac{\partial^2 \bar{u}_x}{\partial x^2} + \frac{\partial^2 \bar{u}_x}{\partial y^2} + \frac{\partial^2 \bar{u}_x}{\partial z^2} \right) \quad (2.4b)$$

$$\frac{\partial \bar{u}_y}{\partial t} + \frac{\partial(\bar{u}_y \bar{u}_x)}{\partial x} + \frac{\partial(\bar{u}_y \bar{u}_y)}{\partial y} + \frac{\partial(\bar{u}_y \bar{u}_z)}{\partial z} = \bar{f}_y - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial y} + \nu_t \left(\frac{\partial^2 \bar{u}_y}{\partial x^2} + \frac{\partial^2 \bar{u}_y}{\partial y^2} + \frac{\partial^2 \bar{u}_y}{\partial z^2} \right) \quad (2.4c)$$

$$\frac{\partial \bar{u}_z}{\partial t} + \frac{\partial(\bar{u}_z \bar{u}_x)}{\partial x} + \frac{\partial(\bar{u}_z \bar{u}_y)}{\partial y} + \frac{\partial(\bar{u}_z \bar{u}_z)}{\partial z} = \bar{f}_z - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial z} + \nu_t \left(\frac{\partial^2 \bar{u}_z}{\partial x^2} + \frac{\partial^2 \bar{u}_z}{\partial y^2} + \frac{\partial^2 \bar{u}_z}{\partial z^2} \right) \quad (2.4d)$$

在河道、湖泊或水库水流中，水平尺度一般远大于垂向尺度。如果垂向加速度与重力加速度相比很小，则可以忽略垂向加速度，流速等水力参数沿垂向的变化常采用其垂向平均值，并假定沿水深方向的动水压强分布符合静水压强分布。三维流动的运动微分方程可简化为：

$$\frac{\partial \bar{u}_x}{\partial x} + \frac{\partial \bar{u}_y}{\partial y} + \frac{\partial \bar{u}_z}{\partial z} = 0 \quad (2.5a)$$

$$\frac{\partial \bar{u}_x}{\partial t} + \frac{\partial(\bar{u}_x \bar{u}_x)}{\partial x} + \frac{\partial(\bar{u}_x \bar{u}_y)}{\partial y} + \frac{\partial(\bar{u}_x \bar{u}_z)}{\partial z} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} + \nu_t \left(\frac{\partial^2 \bar{u}_x}{\partial x^2} + \frac{\partial^2 \bar{u}_x}{\partial y^2} + \frac{\partial^2 \bar{u}_x}{\partial z^2} \right) \quad (2.5b)$$

$$\frac{\partial \bar{u}_y}{\partial t} + \frac{\partial(\bar{u}_y \bar{u}_x)}{\partial x} + \frac{\partial(\bar{u}_y \bar{u}_y)}{\partial y} + \frac{\partial(\bar{u}_y \bar{u}_z)}{\partial z} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial y} + \nu_t \left(\frac{\partial^2 \bar{u}_y}{\partial x^2} + \frac{\partial^2 \bar{u}_y}{\partial y^2} + \frac{\partial^2 \bar{u}_y}{\partial z^2} \right) \quad (2.5c)$$

$$\frac{\partial \bar{p}}{\partial z} = -\rho g \quad (2.5d)$$

水深积分平均法则

将式(2.5)沿水深积分平均，即可得到沿水深平均的平面二维流动的基本方程。在沿水深积分平均过程中，采用以下定义和公式：

(1) 定义水深为

$$H = z_s - z_b \quad (2.6)$$

式中， H 为水深， $z_s = z_s(x, y, t)$ 、 $z_b = z_b(x, y, t)$ 分别为某一基准面下的水面高程和河床高程（见图）

图 2.1: 水位基准示意图

(2) 定义沿水深平均流速 U_i 为

$$U_i = \frac{1}{H} \int_{z_b}^{z_s} \bar{u}_i dz \quad (2.7)$$

式中, 下标 i 取 1, 2 和 3 分别对应 x , y 和 z 方向的速度分量。

(3) 莱布尼兹公式

$$\frac{\partial}{\partial x_i} \int_a^b f dz = \int_a^b \frac{\partial f}{\partial x_i} dz + f|_b \frac{\partial b}{\partial x_i} - f|_a \frac{\partial a}{\partial x_i} \quad (2.8)$$

式中, $x_i = x, y, z$ 。 a, b 和 f 都是 x_i 的函数。

(4) 自由表面及河床底部运动学条件为:

$$\bar{u}_z|_{z_s} = \frac{Dz_s}{Dt} = \frac{\partial z_s}{\partial t} + \bar{u}_x|_{z_s} \frac{\partial z_s}{\partial x} + \bar{u}_y|_{z_s} \frac{\partial z_s}{\partial y} \quad (2.9)$$

$$\bar{u}_z|_{z_b} = \frac{Dz_b}{Dt} = \frac{\partial z_b}{\partial t} + \bar{u}_x|_{z_b} \frac{\partial z_b}{\partial x} + \bar{u}_y|_{z_b} \frac{\partial z_b}{\partial y} \quad (2.10)$$

2.3.2 沿水深平均的连续性方程

采用上述定义和公式对连续性方程(2.5a)沿水深积分平均得:

$$\int_{z_b}^{z_s} \left(\frac{\partial \bar{u}_x}{\partial x} + \frac{\partial \bar{u}_y}{\partial y} + \frac{\partial \bar{u}_z}{\partial z} \right) dz = 0 \quad (2.11)$$

根据式(2.8), 式(2.11)中前两项分别可写成

$$\int_{z_b}^{z_s} \frac{\partial \bar{u}_x}{\partial x} dz = \frac{\partial}{\partial x} \int_{z_b}^{z_s} \bar{u}_x dz - \bar{u}_x|_{z_s} \frac{\partial z_s}{\partial x} + \bar{u}_x|_{z_b} \frac{\partial z_b}{\partial x} \quad (2.12)$$

$$\int_{z_b}^{z_s} \frac{\partial \bar{u}_y}{\partial y} dz = \frac{\partial}{\partial y} \int_{z_b}^{z_s} \bar{u}_y dz - \bar{u}_y|_{z_s} \frac{\partial z_s}{\partial y} + \bar{u}_y|_{z_b} \frac{\partial z_b}{\partial y} \quad (2.13)$$

式(2.11)中最后一项

$$\int_{z_b}^{z_s} \frac{\partial \bar{u}_z}{\partial z} dz = \bar{u}_z|_{z_s} - \bar{u}_z|_{z_b} \quad (2.14)$$

将式(2.12), (2.13)和(2.14) 代入(2.11), 并利用自由表面及河床底部运动学条件式(2.9) 和(2.10), 可得

$$\begin{aligned} & \int_{z_b}^{z_s} \left(\frac{\partial \bar{u}_x}{\partial x} + \frac{\partial \bar{u}_y}{\partial y} + \frac{\partial \bar{u}_z}{\partial z} \right) dz \\ &= \frac{\partial}{\partial x} \int_{z_b}^{z_s} \bar{u}_x dz - \bar{u}_x|_{z_s} \frac{\partial z_s}{\partial x} + \bar{u}_x|_{z_b} \frac{\partial z_b}{\partial x} + \\ & \quad \frac{\partial}{\partial y} \int_{z_b}^{z_s} \bar{u}_y dz - \bar{u}_y|_{z_s} \frac{\partial z_s}{\partial y} + \bar{u}_y|_{z_b} \frac{\partial z_b}{\partial y} + \\ & \quad \bar{u}_z|_{z_s} - \bar{u}_z|_{z_b} \\ &= \frac{\partial H U_x}{\partial x} + \frac{\partial H U_y}{\partial y} + \frac{\partial z_s}{\partial t} - \frac{\partial z_b}{\partial t} = 0 \end{aligned}$$

最后得

$$\frac{\partial H}{\partial t} + \frac{\partial HU_x}{\partial x} + \frac{\partial HU_y}{\partial y} = 0 \quad (2.15)$$

也可以写成

$$\frac{\partial H}{\partial t} + \nabla \cdot (H\mathbf{U}) = 0 \quad (2.16)$$

2.3.3 沿水深平均的运动方程

以 x 方向为例, 式(2.5b)沿水深积分为

$$\int_{z_b}^{z_s} \left[\frac{\partial \bar{u}_x}{\partial t} + \frac{\partial (\bar{u}_x \bar{u}_x)}{\partial x} + \frac{\partial (\bar{u}_x \bar{u}_y)}{\partial y} + \frac{\partial (\bar{u}_x \bar{u}_z)}{\partial z} + \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} - \nu_t \left(\frac{\partial^2 \bar{u}_x}{\partial x^2} + \frac{\partial^2 \bar{u}_x}{\partial y^2} + \frac{\partial^2 \bar{u}_x}{\partial z^2} \right) \right] dz = 0 \quad (2.17)$$

式(2.17)中包含了非恒定流项积分、对流项积分、压力项积分和阻力项积分。接下来分项讨论。

(1) 非恒定流项积分

$$\begin{aligned} \int_{z_0}^{z_s} \frac{\partial \bar{u}_x}{\partial t} dz &= \frac{\partial}{\partial t} \int_{z_0}^{z_s} \bar{u}_x dz - \bar{u}_x|_{z_s} \frac{\partial z_s}{\partial t} + \bar{u}_x|_{z_b} \frac{\partial z_b}{\partial t} \\ &= \frac{\partial HU_x}{\partial t} - \bar{u}_x|_{z_s} \frac{\partial z_s}{\partial t} + \bar{u}_x|_{z_b} \frac{\partial z_b}{\partial t} \end{aligned} \quad (2.18)$$

(2) 对流项积分

首先将时均流速按照式(2.19)进行分解

$$\begin{aligned} \bar{u}_x &= U_x + \Delta \bar{u}_x \\ \bar{u}_y &= U_y + \Delta \bar{u}_y \end{aligned} \quad (2.19)$$

式中, $\Delta \bar{u}_x$ 和 $\Delta \bar{u}_y$ 分别为 x 和 y 的时均流速与垂线平均流速的差值。

对流项中第一项的积分

$$\int_{z_0}^{z_s} \frac{\partial \bar{u}_x \bar{u}_x}{\partial x} dz = \frac{\partial}{\partial x} \int_{z_0}^{z_s} \bar{u}_x \bar{u}_x dz - \bar{u}_x \bar{u}_x|_{z_s} \frac{\partial z_s}{\partial x} + \bar{u}_x \bar{u}_x|_{z_b} \frac{\partial z_b}{\partial x} \quad (2.20)$$

式中

$$\begin{aligned} \int_{z_0}^{z_s} \bar{u}_x \bar{u}_x dz &= \int_{z_0}^{z_s} (U_x + \Delta \bar{u}_x)(U_x + \Delta \bar{u}_x) dz \\ &= \int_{z_0}^{z_s} (U_x U_x + 2U_x \Delta \bar{u}_x + \Delta \bar{u}_x \Delta \bar{u}_x) dz \\ &= HU_x U_x + \int_{z_0}^{z_s} \Delta \bar{u}_x \Delta \bar{u}_x dz \\ &= \beta_{xx} HU_x U_x \end{aligned}$$

其中

$$\beta_{xx} = 1 + \frac{\int_{z_0}^{z_s} \Delta \bar{u}_x \Delta \bar{u}_x dz}{HU_x U_x} \quad (2.21)$$

是由于流速沿垂线分布不均匀而引入的修正系数, 类似于水力学中的动量修正系数。 β_{xx} 的取值一般在 1.02 与 1.05 之间, 可近似取为 1.0。因此

$$\int_{z_0}^{z_s} \frac{\partial \bar{u}_x \bar{u}_x}{\partial x} dz = \frac{\partial HU_x U_x}{\partial x} - \bar{u}_x \bar{u}_x|_{z_s} \frac{\partial z_s}{\partial x} + \bar{u}_x \bar{u}_x|_{z_b} \frac{\partial z_b}{\partial x} \quad (2.22)$$

同理, 对流项的第二项可写为

$$\int_{z_0}^{z_s} \frac{\partial \bar{u}_x \bar{u}_y}{\partial y} dz = \frac{\partial HU_x U_y}{\partial x} - \bar{u}_x \bar{u}_y|_{z_s} \frac{\partial z_s}{\partial y} + \bar{u}_x \bar{u}_y|_{z_b} \frac{\partial z_b}{\partial y} \quad (2.23)$$

对流项的第三项

$$\int_{z_0}^{z_s} \frac{\partial \bar{u}_x \bar{u}_z}{\partial z} dz = \bar{u}_x \bar{u}_z \Big|_{z_s} - \bar{u}_x \bar{u}_z \Big|_{z_b} \quad (2.24)$$

将非恒定流项和对流项积分相加，并利用自由表面和河床底部运动学条件可得：

$$\int_{z_b}^{z_s} \left[\frac{\partial \bar{u}_x}{\partial t} + \frac{\partial(\bar{u}_x \bar{u}_x)}{\partial x} + \frac{\partial(\bar{u}_x \bar{u}_y)}{\partial y} + \frac{\partial(\bar{u}_x \bar{u}_z)}{\partial z} \right] dz = \frac{\partial H U_x}{\partial t} + \frac{\partial H U_x U_x}{\partial x} + \frac{\partial H U_x U_y}{\partial y} \quad (2.25)$$

(3) 压力项积分

$$\begin{aligned} \int_{z_b}^{z_s} \frac{\partial \bar{p}}{\partial x} dz &= \frac{\partial}{\partial x} \int_{z_0}^{z_s} \bar{p} dz - \bar{p} \Big|_{z_s} \frac{\partial z_s}{\partial x} + \bar{p} \Big|_{z_b} \frac{\partial z_b}{\partial x} \\ &= \frac{\partial}{\partial x} \int_{z_0}^{z_s} \rho g(z_s - z) dz - \rho g(z_s - z) \Big|_{z_s} \frac{\partial z_s}{\partial x} + \rho g(z_s - z) \Big|_{z_b} \frac{\partial z_b}{\partial x} \\ &= \rho g H \frac{\partial H}{\partial x} + \rho g H \frac{\partial z_b}{\partial x} = \rho g H \frac{\partial z_s}{\partial x} \end{aligned} \quad (2.26)$$

(4) 阻力项积分

$$\begin{aligned} \int_{z_b}^{z_s} \frac{\partial^2 \bar{u}_x}{\partial x^2} dz &= \int_{z_b}^{z_s} \frac{\partial}{\partial x} \left(\frac{\partial \bar{u}_x}{\partial x} \right) dz = \frac{\partial}{\partial x} \int_{z_0}^{z_s} \frac{\partial \bar{u}_x}{\partial x} dz - \frac{\partial \bar{u}_x}{\partial x} \Big|_{z_s} \frac{\partial z_s}{\partial x} + \frac{\partial \bar{u}_x}{\partial x} \Big|_{z_b} \frac{\partial z_b}{\partial x} \\ &= \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \int_{z_0}^{z_s} \bar{u}_x dz - \bar{u}_x \Big|_{z_s} \frac{\partial z_s}{\partial x} + \bar{u}_x \Big|_{z_b} \frac{\partial z_b}{\partial x} \right) - \frac{\partial \bar{u}_x}{\partial x} \Big|_{z_s} \frac{\partial z_s}{\partial x} + \frac{\partial \bar{u}_x}{\partial x} \Big|_{z_b} \frac{\partial z_b}{\partial x} \\ &= \frac{\partial^2 H U_x}{\partial x^2} + \frac{\partial}{\partial x} \left(-\bar{u}_x \Big|_{z_s} \frac{\partial z_s}{\partial x} + \bar{u}_x \Big|_{z_b} \frac{\partial z_b}{\partial x} \right) - \frac{\partial \bar{u}_x}{\partial x} \Big|_{z_s} \frac{\partial z_s}{\partial x} + \frac{\partial \bar{u}_x}{\partial x} \Big|_{z_b} \frac{\partial z_b}{\partial x} \end{aligned} \quad (2.27)$$

类似地

$$\int_{z_b}^{z_s} \frac{\partial^2 \bar{u}_x}{\partial y^2} dz = \frac{\partial^2 H U_x}{\partial y^2} + \frac{\partial}{\partial y} \left(-\bar{u}_x \Big|_{z_s} \frac{\partial z_s}{\partial y} + \bar{u}_x \Big|_{z_b} \frac{\partial z_b}{\partial y} \right) - \frac{\partial \bar{u}_x}{\partial y} \Big|_{z_s} \frac{\partial z_s}{\partial y} + \frac{\partial \bar{u}_x}{\partial y} \Big|_{z_b} \frac{\partial z_b}{\partial y} \quad (2.28)$$

$$\int_{z_b}^{z_s} \frac{\partial^2 \bar{u}_x}{\partial z^2} dz = \frac{\partial \bar{u}_x}{\partial z} \Big|_{z_s} - \frac{\partial \bar{u}_x}{\partial z} \Big|_{z_b} \quad (2.29)$$

将式(2.27)至(2.29)相加，得

$$\begin{aligned} &\int_{z_b}^{z_s} \nu_t \left(\frac{\partial^2 \bar{u}_x}{\partial x^2} + \frac{\partial^2 \bar{u}_x}{\partial y^2} + \frac{\partial^2 \bar{u}_x}{\partial z^2} \right) dz = \\ &\nu_t \left(\frac{\partial^2 H U_x}{\partial x^2} + \frac{\partial^2 H U_x}{\partial y^2} \right) \\ &- \nu_t \left[\frac{\partial}{\partial x} \left(\frac{\partial z_s}{\partial x} \bar{u}_x \Big|_{z_s} \right) + \frac{\partial}{\partial y} \left(\frac{\partial z_s}{\partial y} \bar{u}_x \Big|_{z_s} \right) + \frac{\partial \bar{u}_x}{\partial x} \Big|_{z_s} \frac{\partial z_s}{\partial x} + \frac{\partial \bar{u}_x}{\partial y} \Big|_{z_s} \frac{\partial z_s}{\partial y} - \frac{\partial \bar{u}_x}{\partial z} \Big|_{z_s} \right] \\ &+ \nu_t \left[\frac{\partial}{\partial x} \left(\frac{\partial z_b}{\partial x} \bar{u}_x \Big|_{z_b} \right) + \frac{\partial}{\partial y} \left(\frac{\partial z_b}{\partial y} \bar{u}_x \Big|_{z_b} \right) + \frac{\partial \bar{u}_x}{\partial x} \Big|_{z_b} \frac{\partial z_b}{\partial x} + \frac{\partial \bar{u}_x}{\partial y} \Big|_{z_b} \frac{\partial z_b}{\partial y} - \frac{\partial \bar{u}_x}{\partial z} \Big|_{z_b} \right] \end{aligned} \quad (2.30)$$

式(2.30)中右边后两项分别为由底部床面阻力和自由表面风阻力引起的阻力项，通常可以式(2.31)表示：

$$g \frac{n^2 U_x \sqrt{U_x^2 + U_y^2}}{H^{1/3}} - C_w \frac{\rho_a}{\rho} \omega^2 \cos \beta \quad (2.31)$$

式中， C_w 为无因此风应力系数， ρ_a 为空气密度， ω 为风速， β 为风向与 x 方向的夹角。最后， x 方向的运动方程为

$$\begin{aligned} &\frac{\partial H U_x}{\partial t} + \frac{\partial H U_x U_x}{\partial x} + \frac{\partial H U_x U_y}{\partial y} = \\ &- g H \frac{\partial z_s}{\partial x} - g \frac{n^2 U_x \sqrt{U_x^2 + U_y^2}}{H^{1/3}} + \nu_t \left(\frac{\partial^2 H U_x}{\partial x^2} + \frac{\partial^2 H U_x}{\partial y^2} \right) + C_w \frac{\rho_a}{\rho} \omega^2 \cos \beta \end{aligned} \quad (2.32)$$

同理可得, y 方向运动方程为

$$\begin{aligned} \frac{\partial HU_y}{\partial t} + \frac{\partial HU_x U_y}{\partial x} + \frac{\partial HU_y U_y}{\partial y} = \\ -gH \frac{\partial z_s}{\partial y} - g \frac{n^2 U_y \sqrt{U_x^2 + U_y^2}}{H^{1/3}} + \nu_t \left(\frac{\partial^2 HU_y}{\partial x^2} + \frac{\partial^2 HU_y}{\partial y^2} \right) + C_w \frac{\rho_a}{\rho} \omega^2 \sin \beta \end{aligned} \quad (2.33)$$

当自由表面风应力影响较小时, 风应力项可以忽略。此外, 当模拟区域尺度较大, 还要考虑地球自转的影响, 可在方程(2.32)和(2.33)右边分别加入科氏力。

$$\begin{aligned} f_x &= 2\omega \sin \varphi U_x \\ f_y &= 2\omega \sin \varphi U_y \end{aligned} \quad (2.34)$$

式中, ω 为地球自转角速度, φ 为模拟区域所处纬度。

根据推导过程中所采用的假定条件, 在使用二维浅水方程时应注意以下问题:

1. 方程推导中引用了牛顿流体所满足的本构关系式, 因此上述方程只适用于牛顿流体, 对类似高含沙水流的非牛顿流体不适用。
2. 方程推导中对流体做了均质不可压的假设, 因此上述方程只能在含沙量较小的情况下近似使用, 当含沙量较大时, 应考虑密度变化的影响。
3. 在垂向积分过程中, 略去流速等水力参数沿垂直方向的变化, 并假定沿水深方向的动水压强分布符合静水压强分布。因此所研究问题的水平尺度应远大于垂向尺度, 流速等水力参数沿垂直方向的变化较之沿水平方向的变化要小得多。

2.3.4 二维平面浅水方程不同形式

忽略风应力和科氏力的二维平面浅水方程为

$$\begin{aligned} \frac{\partial H}{\partial t} + \frac{\partial HU_x}{\partial x} + \frac{\partial HU_y}{\partial y} &= 0 \\ \frac{\partial HU_x}{\partial t} + \frac{\partial HU_x U_x}{\partial x} + \frac{\partial HU_x U_y}{\partial y} &= -gH \frac{\partial z_s}{\partial x} - g \frac{n^2 U_x \sqrt{U_x^2 + U_y^2}}{H^{1/3}} + \nu_t \left(\frac{\partial^2 HU_x}{\partial x^2} + \frac{\partial^2 HU_x}{\partial y^2} \right) \\ \frac{\partial HU_y}{\partial t} + \frac{\partial HU_x U_y}{\partial x} + \frac{\partial HU_y U_y}{\partial y} &= -gH \frac{\partial z_s}{\partial y} - g \frac{n^2 U_y \sqrt{U_x^2 + U_y^2}}{H^{1/3}} + \nu_t \left(\frac{\partial^2 HU_y}{\partial x^2} + \frac{\partial^2 HU_y}{\partial y^2} \right) \end{aligned} \quad (2.35)$$

当床面阻力占主导作用时, 还可进一步忽略紊动阻力项。

$$\begin{aligned} \frac{\partial H}{\partial t} + \frac{\partial HU_x}{\partial x} + \frac{\partial HU_y}{\partial y} &= 0 \\ \frac{\partial HU_x}{\partial t} + \frac{\partial (HU_x U_x + gH^2/2)}{\partial x} + \frac{\partial HU_x U_y}{\partial y} &= -gH \frac{\partial z_b}{\partial x} - g \frac{n^2 U_x \sqrt{U_x^2 + U_y^2}}{H^{1/3}} \\ \frac{\partial HU_y}{\partial t} + \frac{\partial HU_x U_y}{\partial x} + \frac{\partial (HU_y U_y + gH^2/2)}{\partial y} &= -gH \frac{\partial z_b}{\partial y} - g \frac{n^2 U_y \sqrt{U_x^2 + U_y^2}}{H^{1/3}} \end{aligned} \quad (2.36)$$

此外, 为了书写的简化, 用 h 替代 H , u 替代 U_x , v 替代 U_y , 定义 x 和 y 方向的流量分别为 $q_x = hu$ 和 $q_y = hv$, 上式可写成

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} &= 0 \\ \frac{\partial q_x}{\partial t} + \frac{\partial (uq_x + gh^2/2)}{\partial x} + \frac{\partial vq_y}{\partial y} &= -gh \frac{\partial z_b}{\partial x} - g \frac{n^2 u \sqrt{u^2 + v^2}}{h^{1/3}} \\ \frac{\partial q_y}{\partial t} + \frac{\partial uq_y}{\partial x} + \frac{\partial (vq_y + gh^2/2)}{\partial y} &= -gh \frac{\partial z_b}{\partial y} - g \frac{n^2 v \sqrt{u^2 + v^2}}{h^{1/3}} \end{aligned} \quad (2.37)$$

上式的向量形式为：

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} + \frac{\partial \mathbf{g}}{\partial y} = \mathbf{S}_b + \mathbf{S}_f \quad (2.38)$$

其中， \mathbf{q} 为守恒变量向量， \mathbf{f} 和 \mathbf{g} 分别为 x 和 y 方向的通量向量， \mathbf{S}_b 和 \mathbf{S}_f 分别为河床底坡和阻力源项向量。这些向量的具体元素为：

$$\mathbf{q} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}, \mathbf{f} = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{bmatrix}, \mathbf{g} = \begin{bmatrix} hv \\ huv \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}, \mathbf{S}_b = \begin{bmatrix} 0 \\ -gh \frac{\partial z_b}{\partial x} \\ -gh \frac{\partial z_b}{\partial y} \end{bmatrix}, \mathbf{S}_f = \begin{bmatrix} 0 \\ -g \frac{n^2 u \sqrt{u^2 + v^2}}{h^{1/3}} \\ -g \frac{n^2 v \sqrt{u^2 + v^2}}{h^{1/3}} \end{bmatrix} \quad (2.39)$$

2.4 一维非恒定流基本控制方程

2.4.1 一维连续性方程

如图??所示，在明槽非恒定流中，沿水流流动方向取长为 dx 的微小流段。流段进口 1-1 断面流量为 Q ，在 Δt 时段内，从 1-1 断面进入流段的液体质量为 $\rho Q \Delta t$ 。流段出口 2-2 断面流量为 $Q + \frac{\partial Q}{\partial x} dx$ ，在 Δt 时段内，从 2-2 断面流出的液体质量为 $\rho Q \Delta t + \rho \frac{\partial Q}{\partial x} dx \Delta t$ 。该时段内进出此流段的液体质量差为 $-\rho \frac{\partial Q}{\partial x} dx \Delta t$ 。

时段内的质量差表现为流段内的槽蓄量变化。在起始时刻，流段内的槽蓄量为 $\rho \bar{A} dx$ 。而经过 Δt 时段后，流段内的槽蓄量为 $\rho \left(\bar{A} + \frac{\partial \bar{A}}{\partial t} \Delta t \right) dx$ 。 Δt 时段内，流段内的槽蓄量变化量为 $\rho \frac{\partial \bar{A}}{\partial t} \Delta t dx$ 。其中 \bar{A} 为微小流段的平均过水面积。当流段内过水断面面积变化较小时，可直接用 1-1 断面段面积 A 来替代 \bar{A} 。

因此，根据质量守恒原理，进出该流段的液体质量差等于流段内槽蓄量改变量，即

$$-\rho \frac{\partial Q}{\partial x} dx \Delta t = \rho \frac{\partial A}{\partial t} dx \Delta t$$

化简后得到明槽一维非恒定流连续性方程

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \quad (2.40)$$

如果在该流段内有旁侧入流或出流，且单位长度旁侧入流流量为 q ($q > 0$ 为入流， $q < 0$ 为出流)，考虑旁侧入流得明槽一维非恒定流连续性方程为

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q \quad (2.41)$$

2.4.2 一维运动方程

设坐标轴 x 方向与水流流动方向一致，根据牛顿第二定律建立运动方程。为分析简单起见，首先考虑棱柱体明槽（如图）的情况。

作用在 1-1 断面的所有外力在 x 方向的分力有：

(1) 总动水压力。

设压强分布服从静水压强分布，则作用在 1-1 断面的水压力

$$P = \int_0^h \rho g (h - y) \xi(y) dy \quad (2.42)$$

式中, $\xi(y)$ 为过水断面上距渠底 y 处的宽度。作用于 2-2 断面的水压力为 $P + \frac{\partial P}{\partial x} dx$ 。则沿 x 方向的总动水压力

$$\begin{aligned} \sum P &= P - \left(P + \frac{\partial P}{\partial x} dx \right) \\ &= -\gamma dx \left[\frac{\partial h}{\partial x} \int_0^h z_s(y) dy + \int_0^h (h-y) \frac{\partial z_s(y)}{\partial x} dy \right] \end{aligned} \quad (2.43)$$

因假定明槽为棱柱体明槽, 有 $\frac{\partial \xi(y)}{\partial x} = 0$ 。则有

$$\sum P = -\gamma A \frac{\partial h}{\partial x} dx \quad (2.44)$$

(2) 重力

$$dG_x = dG \sin \alpha = -\gamma A dx \frac{\partial z}{\partial x} \quad (2.45)$$

式中, α 为坐标轴 x 与水平方向的夹角, A 为过水断面面积。

(3) 侧壁面上的阻力

$$dT = \tau_0 \chi dx = \gamma R J \chi dx = \gamma A J dx \quad (2.46)$$

式中, χ 为过水断面湿周, R 为过水断面水力半径, J 为水力坡度, $\tau_0 = \gamma R J$ 为侧壁表面平均切应力。

其次, 由于流速 U 是 x 和 t 的函数, 则水流沿 x 方向的加速度 a_x 为

$$a_x = \frac{dU}{dt} = \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} \quad (2.47)$$

微小流段内的水体质量为 $dm = \rho A dx$ 。

根据牛顿第二定律, 有 $\sum F_x = dma_x$, 即

$$-\gamma A \frac{\partial h}{\partial x} dx - \gamma A \frac{\partial z}{\partial x} dx - \gamma A J dx = \rho A dx \left(\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} \right) \quad (2.48)$$

上式两边同除以 $\gamma A dx$ 并整理得:

$$\frac{\partial z}{\partial x} + \frac{1}{g} \frac{\partial U}{\partial t} + \frac{U}{g} \frac{\partial U}{\partial x} + J = 0 \quad (2.49)$$

式(2.49)即为棱柱体明槽非恒定流运动方程得一般形式。对于非棱柱体明槽(比如河槽向下游缩窄或展宽), 则两岸壁将对微小流段水体作用一附加压力, 该附加压力可表示为

$$Vp' = \int_0^h \left[\rho g (h-y) \frac{\partial \xi(y)}{\partial x} dx \right] dy \quad (2.50)$$

将附加压力代式(2.43)入中, 恰好与该式最后一项抵消。因此对于非棱柱体明槽, 式(2.49)仍适用。

2.4.3 圣维南方程不同形式

连续性方程(2.40)和运动方程(2.49)构成了描述明槽非恒定渐变流的圣维南方程组。在实际应用中, 为了使用方便, 常对式(2.40)和(2.49)进行改写, 得到不同因变量组合的圣维南方程组。

(1) 以水位 z 和流量 Q 为因变量的圣维南方程组

$$\begin{aligned} B \frac{\partial z}{\partial t} + \frac{\partial Q}{\partial x} &= q \\ \frac{\partial Q}{\partial t} + \frac{2Q}{A} \frac{\partial Q}{\partial x} + \left[gA - B \left(\frac{Q}{A} \right)^2 \right] \frac{\partial z}{\partial x} &= \left(\frac{Q}{A} \right)^2 \frac{\partial A}{\partial x} \Big|_z - gA \frac{Q^2}{K^2} \end{aligned} \quad (2.51)$$

(2) 以水位 h 和流量 Q 为因变量的圣维南方程组

$$\begin{aligned} B \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} &= q \\ \frac{\partial Q}{\partial t} + \frac{2Q}{A} \frac{\partial Q}{\partial x} + \left[gA - B \left(\frac{Q}{A} \right)^2 \right] \frac{\partial h}{\partial x} &= \left(\frac{Q}{A} \right)^2 \frac{\partial A}{\partial x} \Big|_h - gA \frac{Q^2}{K^2} \end{aligned} \quad (2.52)$$

(3) 以水深 z 和流量 U 为因变量的圣维南方程组

$$\begin{aligned} \frac{\partial z}{\partial t} + U \frac{\partial z}{\partial x} + \frac{A}{B} \frac{\partial U}{\partial x} &= \frac{1}{B} \left(q - BiU - U \frac{\partial A}{\partial x} \Big|_z \right) \\ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + g \frac{\partial z}{\partial x} &= -g \frac{U^2}{C^2 R} \end{aligned} \quad (2.53)$$

(4) 以水深 h 和流量 U 为因变量的圣维南方程组

$$\begin{aligned} \frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + \frac{A}{B} \frac{\partial U}{\partial x} &= \frac{1}{B} \left(q - U \frac{\partial A}{\partial x} \Big|_h \right) \\ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + g \frac{\partial z}{\partial x} &= g \left(i - \frac{U^2}{C^2 R} \right) \end{aligned} \quad (2.54)$$

(5) 以 A 和 Q 为因变量的圣维南方程组

$$\begin{aligned} \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} &= q \\ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{Q^2}{A} \right) &= -gA \frac{\partial h}{\partial x} - gA \frac{\partial z_b}{\partial x} - g \frac{n^2 |U|}{R^{4/3}} Q \end{aligned} \quad (2.55)$$

或

$$\begin{aligned} \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} &= q \\ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left(\frac{Q^2}{A} \right) &= -gA \frac{\partial Z}{\partial x} - g \frac{n^2 Q |Q|}{AR^{4/3}} \end{aligned} \quad (2.56)$$

2.5 偏微分方程类型和性质

2.5.1 偏微分方程形式

本章前几节所推导的方程都属于偏微分方程，形式各异。对同一模型建立的控制方程也有多种不同的形式。在数值计算中，若控制方程的对流项采用散度的形式来表示，这类方程被称为守恒型的控制方程，否则被称为非恒定形式。

理论上，从微元体角度来看，守恒型微分方程和非守恒型微分方程是等价的，都是同一物理定律的数学表示。但是，数值计算是对有限大小的计算单元进行的，对有限大小的计算体积，两种形式的控制方程有不同的特性。守恒型微分方程允许流动参数在计算单元或控制体内部存在间断；而非守恒型微分方程要求流动参数是可微的。因此，基于守恒型微分方程的数值方法，可以直接用来计算有间断（如激波）的流场，且不用对间断进行任何特殊处理。这类数值方法被称为激波捕捉方法。而基于非守恒型为微分方程的数值方法，一般无法正确的计算有间断的流场。为了处理有间断的流动，这类方法必须与激波装配方法联合使用。简单来说，激波装配方法是把间断从流场中分离出来，当作流场的边界来处理。

2.5.2 控制方程的数学分类及其对数值解的影响

双曲型方程

抛物型方程

椭圆型方程

NS 方程的数学性质

二维浅水方程的数学性质

一维浅水方程的数学性质

第三章 有限差分法

3.1 泰勒展开

3.2 离散基础知识

3.3 差分方程

3.4 相容性、稳定性和收敛性

第四章 有限体积法

4.1 一维热传导方程的有限体积法

4.2 一维对热扩散方程的有限体积法

第五章 一维水动力学问题

5.1 一维恒定流求解

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6.1 网格基础

6.2 二维浅水方程求解