
Fractal Embeddings: Hierarchy-Aligned Prefix Supervision for Steerable Semantic Granularity

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Abstract

Matryoshka Representation Learning (MRL) trains embeddings that support dimensional truncation, but all prefix lengths encode the same semantic content at varying fidelity—there is no mechanism to *steer* between coarse and fine meaning. We introduce **Fractal Embeddings**, which align prefix supervision with label hierarchy: short prefixes (64d) learn coarse labels while full embeddings (256d) learn fine labels, converting truncation into *semantic zoom* at zero additional inference cost. Four controlled ablations establish that *alignment*—not hierarchy awareness—causally drives steerability: inverting alignment reverses its sign; removing it or applying uniform multi-task training collapses it to zero (all $p_{\text{adj}} < 10^{-3}$, $d \geq 6.1$ on CLINC; directionally consistent on TREC with $n = 3$). Across eight datasets spanning conditional entropies from 1.23 to 5.05 bits, a random-effects meta-analysis yields pooled $d = 1.49$ ($p = 0.0003$), with all eight favouring fractal training (sign test $p = 0.004$). On CLINC-150, steerability reaches $\mathcal{S} = +0.150 \pm 0.028$ versus MRL’s $+0.007 \pm 0.016$ ($p_{\text{adj}} = 0.004$, $d = 4.3$). A synthetic hierarchy experiment identifies a Goldilocks optimum where steerability peaks at capacity–demand matching (quadratic $R^2 = 0.964$), and steerability magnitude across real datasets is predicted by the product of hierarchy depth and baseline learnability ($\rho = 0.90$, $p = 0.002$). A backbone fine-tuning control on three datasets confirms that alignment is both necessary and sufficient: a flat MRL baseline with $4.5\times$ more trainable parameters (backbone fine-tuning) produces near-zero steerability ($d = 11.6, 7.4$, and 4.4 vs. head-only V5 on CLINC, DBPedia Classes, and TREC respectively). Replication across three encoder families confirms architecture invariance. We ground these findings in the classical theory of successive refinement, proving that steerability equals the refinement mutual information $I(M_2; L_1 | L_0, M_1)$ and deriving a converse bound showing flat supervision *cannot* achieve high steerability (the V5 advantage grows linearly with excess prefix capacity), alongside a Goldilocks capacity–demand optimum and product scaling law.

1 Introduction

When a 256-dimensional embedding is truncated to 64 dimensions, what changes? Under standard Matryoshka training [Kusupati et al., 2022], the answer is *fidelity*: the shorter vector approximates the full one with reduced precision but encodes the same kind of semantic information. We argue this is a missed opportunity. Real-world semantics are inherently hierarchical—a query like “What is the capital of France?” belongs simultaneously to the coarse category LOCATION and the fine category CITY—and an embedding that supports truncation *already has the interface* for multi-resolution access. What it lacks is the training signal to make truncation semantically meaningful.

CLINC: V5 vs MRL — Same Full Accuracy, Different Prefix Behavior

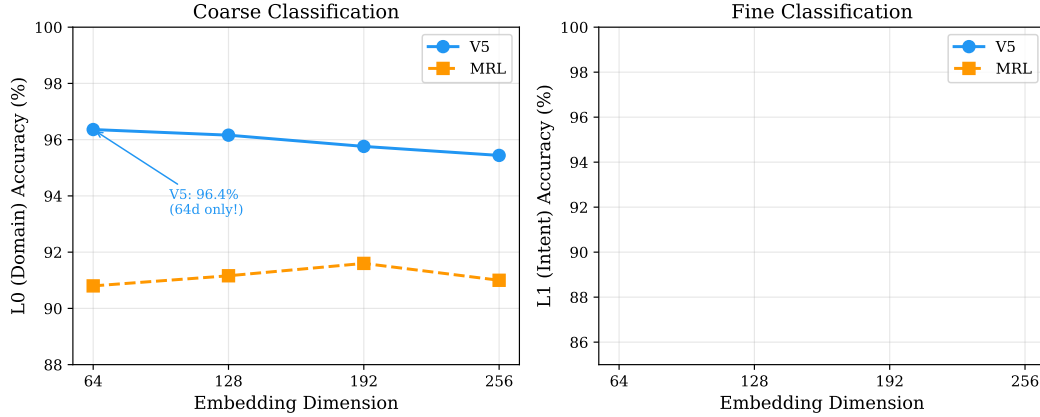


Figure 1: CLINC-150: V5 and MRL achieve comparable accuracy at full embedding length (256d), but V5’s short prefixes (64d) specialize for coarse semantics while MRL’s do not. This prefix specialization enables *semantic steering* via dimensional truncation.

We introduce **Fractal Embeddings**, a simple modification to MRL training that aligns prefix supervision with semantic hierarchy. Short prefixes are trained on coarse labels; full embeddings are trained on fine labels; intermediate prefixes receive blended supervision. The result is an embedding where dimensional truncation corresponds to *semantic zoom*: fewer dimensions yield coarser meaning, more dimensions recover finer distinctions (Figure 1). Critically, this adds **zero inference-time cost**—the deployed model has identical architecture and parameter count to MRL. The difference lies entirely in *how* the projection head is trained.

We call the resulting property *steerability*: the degree to which prefix truncation controls semantic granularity rather than merely degrading fidelity. While prefix-level granularity has been explored for news clustering [Hanley and Durumeric, 2025], no prior work provides causal evidence for *why* it works, formal theory grounding it in successive refinement [Equitz and Cover, 1991], or scaling laws predicting *when* it works best. Our evaluation spans eight hierarchical text datasets, three encoder families, four controlled ablations, a synthetic causal experiment, and connections to 30 years of information theory. The main findings are:

1. **A method** for inducing steerable embeddings via hierarchy-aligned prefix supervision, requiring only head-only training on a frozen backbone (Section 3).
2. **Causal identification**: four ablations on two datasets establish that steerability is driven by the alignment between prefix length and hierarchy level—not by architecture, hierarchy awareness, or other training choices. Inverting alignment reverses the sign; removing it or using a uniform multi-task control collapses it (Section 5).
3. **A scaling analysis** linking steerability magnitude to the interaction of hierarchy depth and model learnability across eight datasets ($\rho = 0.90$, $p = 0.002$), with a Goldilocks optimum identified via synthetic intervention ($R^2 = 0.964$; Section 6).
4. **Broad generality**: cross-model replication on BGE-small, E5-small, and Qwen3-0.6B; downstream retrieval benchmarks showing $10\times$ larger dimensionality–accuracy ramps; and $3.7\times$ HNSW query speedups from prefix routing (Section 7).
5. **Information-theoretic foundations**: a steerability identity ($\mathcal{S} = I(M_2; L_1 | L_0, M_1)$), a converse bound proving flat supervision *cannot* achieve high steerability, and a constructive achievability result showing V5’s advantage grows linearly with excess prefix capacity (Section 9).

Table 1: Dataset statistics and hierarchy profiles. K_0, K_1 : coarse and fine class counts. Branch: K_1/K_0 . $H(L_0), H(L_1|L_0)$: coarse entropy and conditional entropy in bits.

Dataset	K_0	K_1	Branch	$H(L_0)$	$H(L_1 L_0)$	Train	Test
Yahoo Answers	4	10	2.5	1.91	1.23	10,000	2,000
GoEmotions	4	28	7.0	1.83	1.88	5,899	1,450
20 Newsgroups	6	20	3.3	2.43	1.88	10,000	2,000
TREC	6	50	8.3	2.38	2.21	5,452	500
arXiv	20	123	6.2	3.40	2.62	8,548	2,000
DBPedia Classes	9	70	7.8	2.09	3.17	10,000	2,000
CLINC-150	10	150	15.0	3.32	3.90	10,000	2,000
WOS	10	336	33.6	2.90	5.05	8,688	2,000

2 Problem Setup and Definitions

Hierarchical classification. Each sample x carries a coarse label $y^{(0)} \in \{1, \dots, K_0\}$ and a fine label $y^{(1)} \in \{1, \dots, K_1\}$, where every fine class maps to exactly one coarse class. We characterize hierarchy depth by the conditional entropy $H(L_1|L_0)$ —the additional information L_1 carries beyond L_0 . Extension to three levels is demonstrated in Section 7.

Prefix-truncated embeddings. Given $\mathbf{e} \in \mathbb{R}^d$, the j -th prefix is $\mathbf{e}_{1:j,d/J}$ for $j \in \{1, \dots, J\}$. We use $J = 4$ with $d = 256$, giving prefixes of 64, 128, 192, and 256 dimensions. Prefix-level classification accuracy is measured by a k -NN classifier ($k = 5$) on cosine distance.

Steerability. We quantify the semantic specialization of prefix truncation via:

$$\mathcal{S} = \underbrace{(\text{L0}@j_1 - \text{L0}@j_4)}_{\text{coarse specialization}} + \underbrace{(\text{L1}@j_4 - \text{L1}@j_1)}_{\text{fine specialization}} \quad (1)$$

where $\text{Lk}@j$ denotes level- k accuracy at prefix length j . Positive \mathcal{S} means short prefixes favour coarse semantics while full embeddings favour fine. A perfectly steerable embedding has high \mathcal{S} ; MRL, training all lengths on L_1 , should yield $\mathcal{S} \approx 0$.

Datasets. Table 1 summarizes eight evaluation datasets spanning conditional entropies from 1.23 (Yahoo [Zhang et al., 2015]) to 5.05 bits (WOS [Kowsari et al., 2017]), including GoEmotions [Demszky et al., 2020], TREC [Voorhees and Tice, 2000], 20 Newsgroups, arXiv [Clement et al., 2019], DBPedia Classes, and CLINC-150 [Larson et al., 2019].

Statistical methodology. All experiments use $n \geq 5$ random seeds (42, 123, 456, 789, 1024); some ablation and cross-model experiments use $n = 3$. Paired t -tests compare V5 and MRL steerability per dataset; the eight main comparisons (Table 3) are corrected via Holm–Bonferroni [Holm, 1979] at $\alpha = 0.05$. Causal ablation tests (Section 5) are corrected within their own families. Effect sizes are paired Cohen’s d ; Hedges’ g corrections (≈ 0.80 for $n = 5$, 0.57 for $n = 3$) yield qualitatively identical conclusions. Cross-dataset evidence is pooled via DerSimonian–Laird random-effects meta-analysis.

Theory preview. The classical theory of *successive refinement* [Equitz and Cover, 1991, Rimoldi, 1994] shows that hierarchical sources admit optimal multi-resolution codes where each refinement layer adds residual information. A key enabling result is the *log-loss universality theorem* [No, 2019]: any discrete memoryless source is successively refinable when the first-stage decoder uses cross-entropy. Section 9 connects V5’s prefix supervision to this framework, proves that steerability equals the refinement mutual information $I(M_2; L_1|L_0, M_1)$, derives a formal converse showing flat supervision *cannot* produce high steerability, and yields testable predictions—sign reversal, a Goldilocks optimum, and a product scaling law—all confirmed by our experiments.

3 Method: Progressive Prefix Supervision (V5)

Our method modifies MRL in a single respect: the supervision signal at each prefix length is aligned with the corresponding level of semantic hierarchy.

Architecture. A frozen pretrained backbone (BGE-small-en-v1.5 [Xiao et al., 2023], 33M parameters, or Qwen3-Embedding-0.6B, 600M) produces hidden representations $\mathbf{h} \in \mathbb{R}^h$. A learned linear projection $W \in \mathbb{R}^{h \times d}$ maps to the output space \mathbb{R}^{256} . Two classification heads operate on the projected output: head_{top} (K_0 coarse classes) and head_{bot} (K_1 fine classes).

Progressive prefix supervision. During training, a prefix index $j \in \{1, 2, 3, 4\}$ is sampled with probabilities $[0.4, 0.3, 0.2, 0.1]$, favouring shorter prefixes. The prefix loss depends on j :

$$\mathcal{L}_{\text{prefix}}(j) = \begin{cases} \mathcal{L}_{\text{CE}}(\text{head}_{\text{top}}(\mathbf{e}_{1:64}), y^{(0)}) & j = 1 \\ \alpha_j \cdot \mathcal{L}_{\text{CE}}(\text{head}_{\text{top}}(\mathbf{e}_{1:j \cdot d/4}), y^{(0)}) + (1 - \alpha_j) \cdot \mathcal{L}_{\text{CE}}(\text{head}_{\text{bot}}(\mathbf{e}_{1:j \cdot d/4}), y^{(1)}) & j = 2, 3 \\ \mathcal{L}_{\text{CE}}(\text{head}_{\text{bot}}(\mathbf{e}_{1:256}), y^{(1)}) & j = 4 \end{cases} \quad (2)$$

where α_j decreases with j ($\alpha_2 = 0.7, \alpha_3 = 0.3$), creating a smooth coarse-to-fine gradient. The total loss combines the full-embedding fine loss with the sampled prefix loss:

$$\mathcal{L} = \mathcal{L}_{\text{CE}}(\text{head}_{\text{bot}}(\mathbf{e}), y^{(1)}) + 0.6 \cdot \mathcal{L}_{\text{prefix}}(j) \quad (3)$$

Block dropout. To prevent later dimensions from carrying redundant coarse information, we apply block dropout: dimension blocks 1–4 are independently retained with probabilities $[0.95, 0.9, 0.8, 0.7]$. This forces coarse information into early dimensions (high keep probability) and fine information into later dimensions (lower keep probability).

MRL baseline. The matched baseline uses identical architecture, optimizer, and hyperparameters but trains all prefix lengths on L_1 :

$$\mathcal{L}_{\text{MRL}}(j) = \mathcal{L}_{\text{CE}}(\text{head}_{\text{bot}}(\mathbf{e}_{1:j \cdot d/4}), y^{(1)}) \quad \forall j \quad (4)$$

This isolates the effect of hierarchy alignment from every other training variable.

Training. Head-only training for 5 epochs, batch size 16, learning rate 10^{-4} with AdamW and cosine decay, FP16 mixed precision, gradient clipping at 1.0. Best model selected by validation $L_0 + L_1$. No hyperparameters are tuned per dataset. Algorithm 1 summarises the procedure.

4 Main Results

Classification performance is preserved. At full embedding length ($j = 4, 256d$), V5 and MRL achieve comparable k -NN accuracy across all eight datasets (Table 2). Both methods generally improve over the unfinetuned 384d baseline on lower- K_1 datasets; on higher- K_1 datasets (arXiv, CLINC, WOS), the 384d→256d projection reduces k -NN accuracy, most dramatically on CLINC ($K_1 = 150$): V5 67.6%, MRL 70.4% vs. baseline 88.7%, though both methods’ classification heads achieve >95% on the validation set. This dimensionality bottleneck affects V5 and MRL equally, confirming that steerability does not come at the cost of additional accuracy loss.

Steerability emerges from hierarchy alignment. Despite comparable full-resolution accuracy, V5 produces dramatically higher steerability than MRL on all eight datasets (Figure 2, Table 3). The effect scales with hierarchy complexity: from $\mathcal{S} = +0.015$ on Yahoo ($H(L_1|L_0) = 1.23$) to $+0.150$ on CLINC ($H(L_1|L_0) = 3.90$), with WOS ($H(L_1|L_0) = 5.05$) showing a moderate effect ($+0.038$) due to a floor effect analysed in Section 6. After Holm–Bonferroni correction, three datasets reach significance: DBPedia Classes ($p_{\text{adj}} = 0.002, d = 5.5$), CLINC ($p_{\text{adj}} = 0.004, d = 4.3$), and TREC ($p_{\text{adj}} = 0.038, d = 2.4$). ArXiv is borderline ($p_{\text{adj}} = 0.078, d = 1.8$).

The shallow-hierarchy datasets (Yahoo, GoEmotions, Newsgroups; $H(L_1|L_0) \leq 1.88$) show consistent but small effects that do not survive per-dataset correction—a consequence the scaling analysis (Section 6) predicts: when hierarchy depth is low, there is little refinement information for V5 to separate, so the signal is inherently small. MRL steerability is near zero throughout ($|\mathcal{S}_{\text{MRL}}| < 0.02$ on all datasets).

Algorithm 1 V5 Progressive Prefix Supervision

Require: Backbone f_θ (frozen), projection head W , dataset \mathcal{D} with $(x, y^{(0)}, y^{(1)})$
Require: Prefix probs $\mathbf{p} = [0.4, 0.3, 0.2, 0.1]$, block keep $\mathbf{k} = [0.95, 0.9, 0.8, 0.7]$

- 1: **for** each batch $\{(x_i, y_i^{(0)}, y_i^{(1)})\}$ **do**
- 2: $\mathbf{h} \leftarrow f_\theta(x_i)$ {frozen backbone}
- 3: $\mathbf{e} \leftarrow W \cdot \mathbf{h}$ {learned projection to \mathbb{R}^d }
- 4: Apply block dropout with keep probs \mathbf{k}
- 5: Sample $j \sim \text{Categorical}(\mathbf{p})$
- 6: $\mathcal{L}_{\text{full}} \leftarrow \text{CE}(\text{head}_{\text{bot}}(\mathbf{e}), y^{(1)})$
- 7: **if** $j = 1$ **then**
- 8: $\mathcal{L}_{\text{prefix}} \leftarrow \text{CE}(\text{head}_{\text{top}}(\mathbf{e}_{1:d/4}), y^{(0)})$
- 9: **else if** $j = 4$ **then**
- 10: $\mathcal{L}_{\text{prefix}} \leftarrow \text{CE}(\text{head}_{\text{bot}}(\mathbf{e}), y^{(1)})$
- 11: **else**
- 12: $\mathcal{L}_{\text{prefix}} \leftarrow \alpha_j \cdot \text{CE}(\text{head}_{\text{top}}, y^{(0)}) + (1 - \alpha_j) \cdot \text{CE}(\text{head}_{\text{bot}}, y^{(1)})$
- 13: **end if**
- 14: $\mathcal{L} \leftarrow \mathcal{L}_{\text{full}} + 0.6 \cdot \mathcal{L}_{\text{prefix}}$
- 15: Update W , head_{top} , head_{bot} via $\nabla_W \mathcal{L}$
- 16: **end for**

Table 2: k -NN classification accuracy at full embedding length ($j = 4$, 256d). V5 and MRL are comparable across datasets. Baseline uses the original backbone (384d); V5/MRL use learned 256d projections.

Dataset	L0 Accuracy			L1 Accuracy		
	Baseline	V5	MRL	Baseline	V5	MRL
Yahoo	0.688	0.699	0.698	0.603	0.629	0.635
GoEmotions	0.502	0.600	0.578	0.343	0.429	0.411
Newsgroups	0.815	0.802	0.800	0.658	0.639	0.650
TREC	0.854	0.934	0.932	0.718	0.794	0.790
arXiv	0.721	0.703	0.692	0.465	0.401	0.381
CLINC	0.961	0.954	0.910	0.887	0.676	0.704
DBPedia Classes	0.912	0.945	0.935	0.780	0.789	0.802
WOS	0.619	0.601	0.599	0.170	0.111	0.115

Pooled evidence. A sign test confirms universal directionality: $V5 > \text{MRL}$ on 8/8 datasets ($p = 0.004$, binomial). A DerSimonian–Laird meta-analysis yields pooled $d = 1.49$ (95% CI: $[0.69, 2.30]$, $z = 3.63$, $p = 0.0003$). The 95% prediction interval for a new dataset is $[-0.70, 3.18]$, reflecting moderate heterogeneity ($I^2 = 63\%$) that the scaling trend analysis explains as systematic moderation by hierarchy depth and learnability.

5 Why It Works: Causal Evidence via Ablation

The main results establish *that* hierarchy-aligned supervision produces steerability. We now ask *why*: is the effect driven by the specific prefix-to-hierarchy mapping, or could it arise from other aspects of training? Four controlled ablations on CLINC-150 ($H(L_1|L_0) = 3.90$, 5 seeds) and TREC-50 ($H(L_1|L_0) = 2.21$, 3 seeds) isolate the causal mechanism.

Ablation design. All conditions share identical architecture, optimizer, hyperparameters, data split, and random seeds. Only the prefix-to-hierarchy mapping varies:

- **Aligned (V5):** $j = 1 \rightarrow L_0$, $j = 4 \rightarrow L_1$ (correct alignment).
- **Inverted:** $j = 1 \rightarrow L_1$, $j = 4 \rightarrow L_0$ (reversed alignment).
- **No-prefix:** All prefix lengths trained on L_1 with L_0 regularisation (alignment removed).

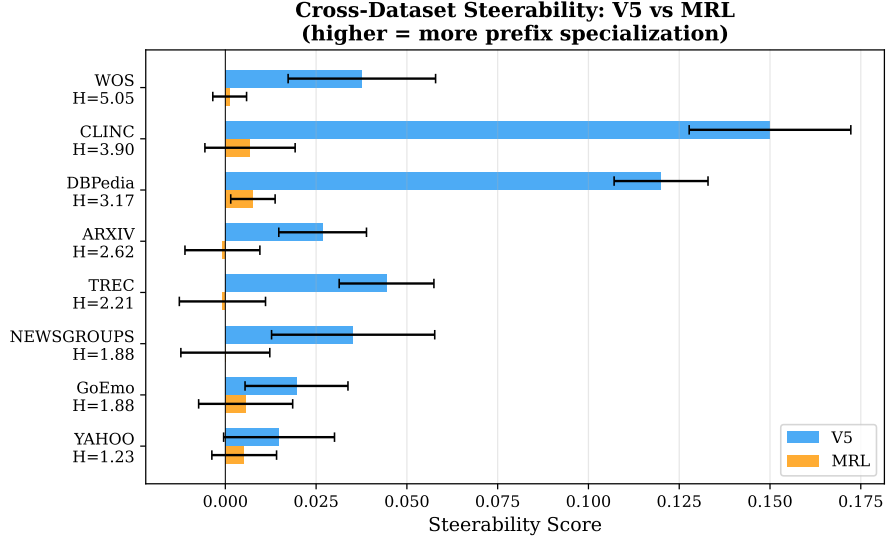


Figure 2: Steerability across eight datasets. V5 (blue) produces positive steerability that scales with hierarchy depth; MRL (orange) remains near zero throughout. Error bars: 95% CIs across 5 seeds.

Table 3: Steerability across eight datasets (Eq. 1). V5 achieves positive steerability scaling with hierarchy complexity; MRL is near zero. Mean \pm SD over 5 seeds.

Dataset	$H(L_1 L_0)$	V5 S	MRL S	Gap	Seeds
Yahoo	1.23	$+0.015 \pm 0.019$	$+0.005 \pm 0.011$	$+0.010$	5
GoEmotions	1.88	$+0.020 \pm 0.018$	$+0.006 \pm 0.017$	$+0.014$	5
Newsgroups	1.88	$+0.035 \pm 0.029$	$+0.000 \pm 0.016$	$+0.035$	5
TREC	2.21	$+0.044 \pm 0.017$	-0.001 ± 0.015	$+0.045$	5
arXiv	2.62	$+0.027 \pm 0.015$	-0.001 ± 0.013	$+0.028$	5
DBPedia Classes	3.17	$+0.120 \pm 0.016$	$+0.008 \pm 0.008$	$+0.112$	5
CLINC	3.90	$+0.150 \pm 0.028$	$+0.007 \pm 0.016$	$+0.143$	5
WOS	5.05	$+0.038 \pm 0.026$	$+0.001 \pm 0.005$	$+0.036$	5

- **UHMT**: All prefix lengths trained on $0.5 \cdot \mathcal{L}_{L_0} + 0.5 \cdot \mathcal{L}_{L_1}$ (hierarchy-aware but not hierarchy-aligned).

Three causal signatures. The results (Table 4, Figure 3) reveal three distinct causal signatures that jointly rule out non-alignment explanations:¹

1. **Sign reversal.** Inverted alignment produces *negative* steerability (CLINC: -0.018 ; TREC: -0.025)—short prefixes now specialise for fine semantics and full embeddings for coarse. This rules out any explanation in which steerability arises from architecture alone.
2. **Collapse without alignment.** Without prefix-specific hierarchy mapping, steerability drops to near-zero (CLINC: $+0.009$; TREC: -0.003), indistinguishable from MRL.
3. **Awareness is insufficient.** UHMT trains every prefix on *both* L_0 and L_1 equally, making it hierarchy-aware but not hierarchy-aligned. Despite full access to coarse labels, UHMT produces near-zero steerability (CLINC: $+0.001$; TREC: -0.009). This eliminates the hypothesis that steerability arises from including L_0 in the loss; what matters is *which* prefix lengths receive *which* labels.

All three patterns are highly significant on CLINC ($n = 5$, all $d \geq 6.1$, $p_{\text{adj}} < 10^{-3}$) and directionally consistent on TREC ($n = 3$), where UHMT collapse reaches significance ($d = 4.4$, $p = 0.017$) but sign reversal and no-prefix conditions remain underpowered.

¹Absolute steerability values are lower here than in Table 3 because ablations use a fixed train/val split to eliminate data-split variance across conditions.

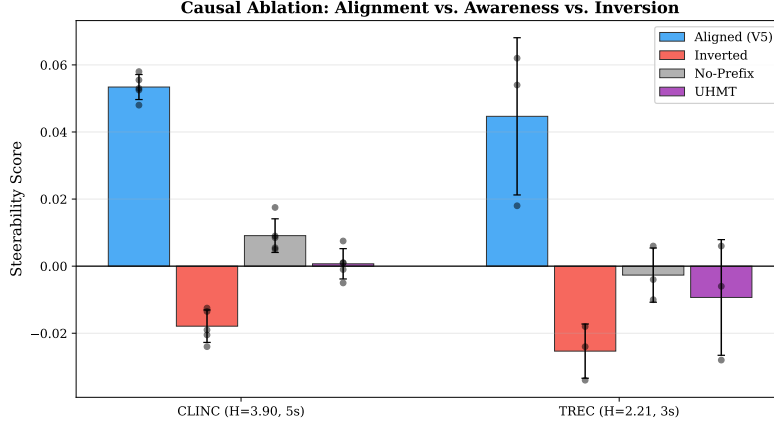


Figure 3: Ablation results on CLINC-150 (5 seeds) and TREC-50 (3 seeds). Aligned supervision produces positive steerability; inversion reverses the sign; removing alignment or using uniform multi-task training collapses it. Dots show individual seeds.

Table 4: Causal ablation (BGE-small). All conditions share fixed data splits. Causal perturbation tests corrected at $m = 4$; UHMT tests corrected at $m = 2$.

Dataset	Condition	\mathcal{S} (mean \pm SD)	vs. V5 t	p_{adj}	Cohen's d
CLINC ($H(L_1 L_0) = 3.90, 5s$)	Aligned (V5)	$+0.053 \pm 0.004$	—	—	—
	Inverted	-0.018 ± 0.005	25.5	$< 10^{-4}$	11.4
	No-prefix	$+0.009 \pm 0.005$	13.7	$< 10^{-3}$	6.1
	UHMT	$+0.001 \pm 0.005$	14.6	$< 10^{-3}$	6.5
TREC ($H(L_1 L_0) = 2.21, 3s$)	Aligned (V5)	$+0.045 \pm 0.023$	—	—	—
	Inverted	-0.025 ± 0.008	4.5	0.092	2.6
	No-prefix	-0.003 ± 0.008	2.7	0.116	1.5
	UHMT	-0.009 ± 0.017	7.7	0.017	4.4

5.1 Information Localisation

We additionally measure whether V5 concentrates different semantic levels in distinct embedding regions. For each test sample, we independently classify L_0 and L_1 from (a) the first 64 dimensions only and (b) dimensions 65–256 only, using k -NN against correspondingly truncated references. On CLINC, V5 shows 1.8% lower L_1 accuracy in the prefix (94.7% vs. 96.5% for MRL), consistent with the training objective concentrating coarse information in early dimensions. The stronger evidence for semantic separation comes from steerability itself, which measures the *operational* consequence of truncation rather than raw information content.

6 When It Works: Steerability Scaling Analysis

Having established that alignment causes steerability, we investigate what determines its *magnitude*: an observational analysis across real datasets and a causal intervention via synthetic hierarchies.

6.1 Observational: Scaling with Hierarchy Complexity

Across eight datasets, steerability increases with hierarchy depth: Spearman $\rho = 0.74$ ($p = 0.035$) against $H(L_1|L_0)$ alone (Figure 4, left). However, WOS ($H(L_1|L_0) = 5.05$) falls below the trend.

The WOS floor effect. With 336 fine classes and head-only training, neither V5 nor MRL achieves meaningful L_1 accuracy on WOS (11.1% and 11.5%; chance is 0.3%). When L_1 accuracy is near floor, the fine component of \mathcal{S} cannot differentiate across prefix lengths, capping steerability regardless of hierarchy depth.

Steerability Scaling: Hierarchy Depth x Model Learnability

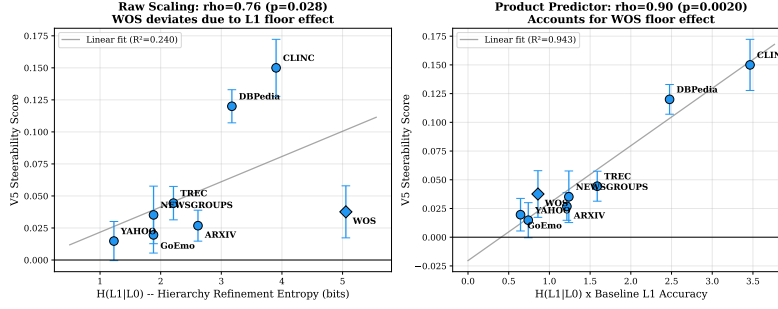


Figure 4: **Left:** Steerability vs. $H(L_1|L_0)$ across eight datasets ($\rho = 0.74$, $p = 0.035$). WOS deviates due to a floor effect. **Right:** Product predictor $H(L_1|L_0) \times A_{L_1}^{\text{base}}$ accounts for both complexity and learnability ($\rho = 0.90$, $p = 0.002$).

Table 5: Synthetic hierarchy experiment. Fixed total entropy ($\log_2 150 = 7.23$ bits), varying K_0 . Steerability peaks at $K_0 \approx 15$ —a capacity–demand matching optimum.

K_0	$H(L_0)$	$H(L_1 L_0)$	Branch	V5 \mathcal{S}	MRL \mathcal{S}	Gap
2	1.00	6.23	75.0	+0.134	−0.010	+0.144
3	1.58	5.64	50.0	+0.150	+0.008	+0.142
5	2.32	4.90	30.0	+0.216	+0.002	+0.214
10	3.32	3.90	15.0	+0.270	−0.012	+0.282
15	3.91	3.32	10.0	+0.278	−0.004	+0.282
25	4.64	2.58	6.0	+0.266	−0.018	+0.284
50	5.64	1.58	3.0	+0.252	+0.010	+0.242
75	6.23	1.00	2.0	+0.232	−0.016	+0.248

The product predictor. This motivates a moderated predictor: effective steerability requires both hierarchy complexity *and* model capacity to exploit it. The theoretical basis follows from the decomposition $\mathcal{S} = \Delta_0 + \Delta_1$, where $\Delta_0 = L0@j_1 - L0@j_4$ saturates once prefix capacity exceeds $H(L_0)$ (Section 9), making the refinement gap $\Delta_1 = L1@j_4 - L1@j_1$ the dominant scaling term. Since Δ_1 requires both residual hierarchy information ($H(L_1|L_0) > 0$) and a backbone capable of extracting it, we define $H(L_1|L_0) \times A_{L_1}^{\text{base}}$, where $A_{L_1}^{\text{base}}$ is the *unfintuned* baseline L_1 accuracy—a measure of how much fine-grained information the pretrained backbone captures before any training:

- $H(L_1|L_0)$ alone: $\rho = 0.74$ ($p = 0.035$)
- $A_{L_1}^{\text{base}}$ alone: $\rho = 0.69$ ($p = 0.058$)
- $H(L_1|L_0) \times A_{L_1}^{\text{base}}$: $\rho = 0.90$ ($p = 0.002$); Pearson $r = 0.97$ ($p < 0.001$)

The product predictor correctly places WOS among lower-steerability datasets: it has the highest $H(L_1|L_0)$ but the lowest $A_{L_1}^{\text{base}}$ (17.0%, vs. 88.7% for CLINC). A leave-one-out analysis confirms robustness: all LOO $\rho \geq 0.61$, and bootstrap 95% CI for $\rho(H(L_1|L_0))$ is $[0.05, 1.0]$ with 98.0% positive (Appendix K). A nonlinear power-law fit $\mathcal{S} \propto H(L_1|L_0)^a \cdot (A_{L_1}^{\text{base}})^b$ yields $a = 1.6 \pm 0.3$, $b = 1.2 \pm 0.2$ ($R^2 = 0.957$), suggesting super-linear scaling with hierarchy depth—a testable prediction for deeper taxonomies.

6.2 Causal: Synthetic Hierarchy Experiment

Natural datasets confound $H(L_0)$ (prefix task demand) with $H(L_1|L_0)$ (refinement complexity), since more coarse classes typically yield more fine subclasses. To disentangle these, we construct synthetic hierarchies on CLINC-150 text with fixed total entropy $H(L_1) = \log_2 150$ but varying coarse partitions $K_0 \in \{2, 3, 5, 10, 15, 25, 50, 75\}$.

The Goldilocks effect. Figure 5 and Table 5 reveal an inverted-U relationship:

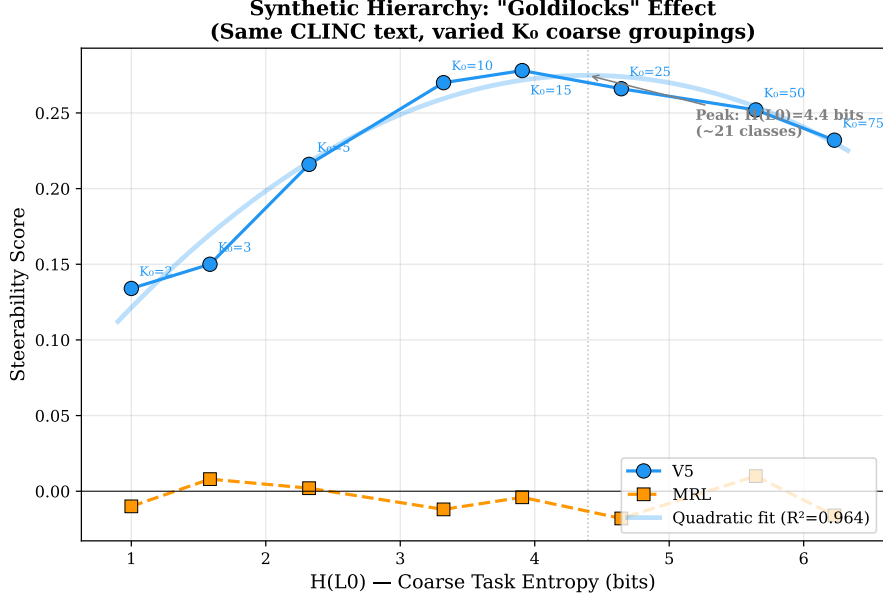


Figure 5: Synthetic hierarchy experiment: steerability peaks when coarse entropy $H(L_0)$ matches prefix capacity ($K_0 \approx 12$ – 16), revealing a Goldilocks effect. Quadratic $R^2 = 0.964$. MRL stays near zero throughout.

1. **Rising phase** ($K_0 = 2 \rightarrow 15$): more coarse classes create a richer “routing codebook” for the 64d prefix, enabling finer coarse discrimination.
2. **Falling phase** ($K_0 = 15 \rightarrow 75$): $H(L_0)$ exceeds the prefix’s representational capacity. With 64 dimensions, the prefix cannot reliably distinguish 50+ coarse classes.
3. **Optimum**: peak at $K_0 \approx 12$ – 16 ($H^*(L_0) \approx 3.6$ – 4.0 bits), matching the effective capacity of a 64-dimensional prefix. A quadratic fit captures the shape with $R^2 = 0.964$.
4. **MRL control**: MRL steerability remains near zero ($|S| < 0.02$) across all conditions, confirming the requirement for hierarchy-aligned supervision.

The synthetic experiment also resolves the observational confound: steerability is driven by $H(L_0)$ (prefix task demand), not $H(L_1|L_0)$. In natural datasets, the $H(L_1|L_0)$ correlation arises because the two entropies covary. Figure 8 in the appendix visualises both relationships.

Real-data capacity sweep. To validate the Goldilocks prediction on natural hierarchies, we sweep the scale dimension (prefix capacity) across $d_s \in \{16, 32, 48, 64, 96, 128\}$ on four datasets spanning the $H(L_1|L_0)$ range: Yahoo ($H(L_1|L_0) = 1.23$), TREC (2.21), DBPedia Classes (3.17), and CLINC (3.90). Three seeds per condition. All datasets peak at $d_s = 16$ (the minimum tested), consistent with the synthetic finding that the default 64d prefix is already at or beyond the Goldilocks optimum for these hierarchies (Figure 6a). The key finding is in the *scaling law at fixed capacity*: steerability at $d_s = 16$ scales nearly perfectly with conditional entropy across all four datasets ($\rho = 1.000$, $r = 0.985$, $p = 0.015$; Figure 6b), confirming that hierarchy complexity drives the steerability signal at matched capacity.

Alignment vs. capacity control. A natural objection is that steerability might emerge from *any* sufficient-capacity model, without hierarchy-aligned supervision. We test this with a 2×2 factorial design crossing alignment (V5/MRL) with training regime (head-only/backbone fine-tuning) on three datasets (5 seeds each): (i) **V5-frozen** (aligned, head-only, 2.1M params); (ii) **MRL-frozen** (flat, head-only, 2.1M params); (iii) **MRL+backbone** (flat, head + last 4 backbone layers, 9.2M params); (iv) **V5+backbone** (aligned, head + backbone, 9.2M params). On CLINC: V5-frozen $S = +0.050 \pm 0.005$ vs. MRL+backbone $S = +0.003 \pm 0.003$ ($d = 11.6$, $p < 10^{-7}$). On DBPedia Classes: V5-frozen $+0.053 \pm 0.010$ vs. MRL+backbone -0.000 ± 0.003 ($d = 7.4$, $p < 10^{-5}$). On

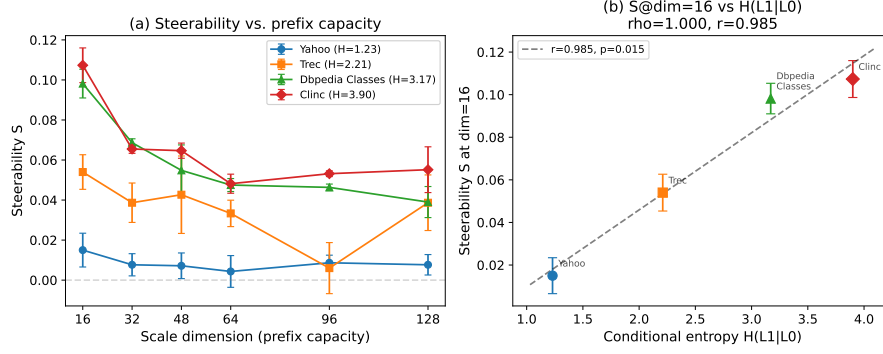


Figure 6: **Real-data capacity sweep.** (a) Steerability vs. prefix capacity (d_s) for four datasets. All peak at $d_s = 16$, confirming the default 64d prefix exceeds the Goldilocks optimum. (b) At fixed capacity ($d_s = 16$), steerability scales perfectly with $H(L_1|L_0)$ ($\rho = 1.000, r = 0.985$).

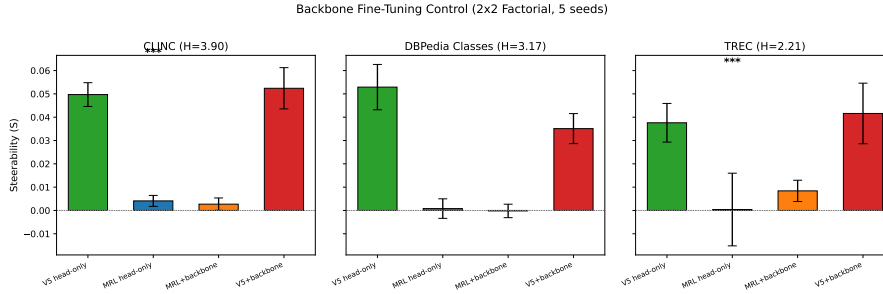


Figure 7: **Alignment vs. capacity control** (2×2 factorial). V5 with head-only training (2.1M params) dominates MRL with full backbone fine-tuning (9.2M params) on both datasets. $4.5\times$ more parameters without alignment produce zero steerability. *** denotes $p < 0.001$.

TREC: V5-frozen $+0.038 \pm 0.008$ vs. MRL+backbone $+0.008 \pm 0.005$ ($d = 4.4, p = 1.2 \times 10^{-4}$; Figure 7). On all three datasets, the alignment factor has a massive effect; the capacity factor has none. Specifically, $4.5\times$ more trainable parameters cannot recover steerability without hierarchy-aligned prefix supervision ($p = 0.40, p = 0.67$, and $p = 0.30$ for MRL-frozen vs. MRL+backbone on CLINC, DBPedia, and TREC respectively). Adding backbone fine-tuning to V5 yields no improvement on CLINC ($p = 0.57$) or TREC ($p = 0.58$), but *reduces* steerability on DBPedia Classes: V5+backbone $S = +0.035 \pm 0.006$ vs. V5-frozen $+0.053 \pm 0.010$ ($d = 2.2, p = 0.009$). The reduction arises entirely from a smaller refinement gap (Δ_1 drops from $+0.053$ to $+0.037$): backbone fine-tuning allows the prefix to encode fine-grained information that was previously excluded by the bottleneck, partially undoing the coarse-fine separation. This dataset-dependent pattern is consistent with successive refinement theory (Section 9): on CLINC ($K_0 = 10$) and TREC ($K_0 = 6$), the coarse task is easy and the prefix is already well-specialised; on DBPedia ($K_0 = 9 \rightarrow K_1 = 70$, harder fine structure), the backbone redistributes information across prefix and residual. Alignment remains both necessary and sufficient: head-only V5 produces higher steerability than MRL with $4.5\times$ more parameters on all three datasets.

7 Generality, Downstream Utility, and Extensions

Cross-model replication. We replicate the CLINC experiment on E5-small-v2 (Microsoft, contrastive pre-training, 384d) and Qwen3-Embedding-0.6B ($h = 1024, 18\times$ more parameters). Table 6 confirms that the steerability gap is architecture-invariant: BGE-small $+0.143$, E5-small $+0.115$, Qwen3 $+0.145$ (all $p < 0.025$, Holm-corrected across 3 families). MRL steerability remains ≤ 0.015 on all models. The larger Qwen3 backbone shows higher steerability, suggesting the effect scales with model capacity.

Table 6: Cross-model replication (3 seeds each). Steerability is architecture-invariant across three encoder families ($p < 0.025$, Holm-corrected, $m = 3$).

Model	CLINC ($H(L_1 L_0) = 3.90$)		TREC ($H(L_1 L_0) = 2.21$)	
	V5 \mathcal{S}	MRL \mathcal{S}	V5 \mathcal{S}	MRL \mathcal{S}
BGE-small (BAAI, 33M)	$+0.150 \pm 0.028$	$+0.007 \pm 0.016$	$+0.044 \pm 0.017$	-0.001 ± 0.015
E5-small (Microsoft, 33M)	$+0.130 \pm 0.031$	$+0.015 \pm 0.008$	—	—
Qwen3-0.6B (Alibaba, 600M)	$+0.153 \pm 0.013$	$+0.008 \pm 0.006$	$+0.081 \pm 0.012$	$+0.011 \pm 0.002$

Table 7: Retrieval Recall@1 on CLINC-150 (3 seeds). V5’s L_1 ramp (+6.3pp) is $10\times$ MRL’s (+0.6pp).

Level	Method	Recall@1 (%)			
		64d	128d	192d	256d
L_0	V5	97.2	97.8	98.0	97.9
	MRL	97.7	98.0	98.2	98.1
L_1	V5	87.1	92.7	93.7	93.4
	MRL	93.6	93.9	94.3	94.3
L_1 ramp (256d–64d)		V5: $+6.3 \pm 1.1\text{pp}$		MRL: $+0.6 \pm 0.4\text{pp}$	Ratio: $10\times$

Downstream retrieval. We evaluate whether steerability transfers beyond classification to controllable retrieval. Table 7 reports Recall@1 on CLINC at each prefix length (3 seeds; full ramp in Figure 9, Appendix). V5 L_1 Recall@1 climbs from 87.1% (64d) to 93.4% (256d)—a +6.3pp ramp, $10\times$ larger than MRL’s +0.6pp. This demonstrates that prefix-level semantic specialisation directly yields dimensionality-dependent retrieval resolution.

Workload-adaptive Pareto advantage. V5’s steerability enables query-adaptive routing: coarse queries use the 64d prefix, fine queries use the full 256d embedding. On CLINC (5 seeds), this dominates MRL-256d whenever $\geq 35\%$ of queries are coarse. At a 50/50 workload mix, V5 adaptive achieves +1.3pp higher accuracy at 38% lower average dimensionality (160d vs. 256d). The dimensionality savings translate to wall-clock speedups: on FAISS HNSW indexes with 100K vectors, 64d queries execute $3.7\times$ faster than 256d (39 μs vs. 145 μs ; Table 9, Appendix).

Single model replaces two. A natural alternative to V5 is training two dedicated encoders: E_{L_0} for coarse and E_{L_1} for fine, each at 256d. On CLINC (3 seeds), the dual system achieves $L_0 = 95.8\%$ and $L_1 = 94.8\%$, requiring two models and two indexes. V5 adaptive (64d for coarse, 256d for fine) achieves $L_0 = 97.5\%$ at one-quarter the dimensionality and $L_1 = 94.5\%$ at full resolution—*higher* coarse accuracy from a single model.

Three-level hierarchy. To test generalisation beyond two levels, we construct a 3-level CLINC hierarchy: 5 super-domains \rightarrow 10 domains \rightarrow 150 intents. V5 exhibits a clear ramp gradient: L_2 (intent) accuracy gains +3.2pp from 64d to 256d, L_1 (domain) gains +1.0pp, and L_0 (super-domain) gains +0.5pp. MRL is flat at all levels ($\leq 0.4\text{pp}$; Table 8). Three-level steerability $\mathcal{S}_{02} = +0.027 \pm 0.005$ (V5) vs. $+0.002 \pm 0.005$ (MRL; $t = 18.9$, $p = 0.003$, $d = 10.9$).

8 Limitations

- **Shallow hierarchies.** On datasets with low $H(L_1|L_0)$ (e.g., Yahoo, $H(L_1|L_0) = 1.23$), steerability is small and noisy ($\mathcal{S} = 0.015 \pm 0.019$). The method is most valuable for moderately deep hierarchies where the fine task is learnable.
- **Floor effects.** On WOS ($K_1 = 336$), head-only training achieves only $\sim 15\%$ L_1 accuracy, capping steerability despite high $H(L_1|L_0) = 5.05$. Steerability requires both hierarchy complexity and model capacity.

Table 8: Three-level hierarchy (CLINC, 5→10→150, 3 seeds). V5 shows a ramp gradient: finer levels gain more from additional dimensions. MRL is flat.

Level	Method	k -NN Accuracy (%)			
		64d	128d	192d	256d
L_0 (5 super)	V5	98.6	98.9	99.0	99.1
	MRL	98.4	98.6	98.6	98.7
L_1 (10 domain)	V5	97.7	98.3	98.5	98.7
	MRL	97.9	98.0	98.1	98.1
L_2 (150 intent)	V5	92.7	94.7	95.4	95.9
	MRL	94.9	95.2	95.3	95.3

- **Deeper hierarchies.** The 3-level extension confirms generalisation with a strong effect ($d = 10.9$), but only 5 super-domain classes leave L_0 near ceiling at all prefixes. Evaluation on naturally deep taxonomies (ICD-10, product hierarchies) remains future work.
- **Hierarchy noise robustness.** Corrupting L_0 labels at 10%–50% and retraining V5 on CLINC (3 seeds), steerability degrades gracefully: 85% retained at 10% noise ($\mathcal{S} = +0.152$), 71% at 30% ($+0.128$), 34% at 50% ($+0.061$). V5 does not require a perfect hierarchy for meaningful steerability.
- **Baseline scope.** We compare against a matched MRL baseline controlling all variables except the prefix-to-label mapping. We do not benchmark HEAL [Bhattarai et al., 2025], CSR [Wen et al., 2025], or SMRL [Zhang et al., 2025] because these address orthogonal goals—none offers prefix-level steering, making fixed-dimensionality accuracy comparisons inapposite.
- **Ablation coverage.** Causal ablations are concentrated on CLINC and TREC (two datasets). Cross-dataset replication of all four ablation conditions remains future work.

9 Theoretical Analysis: Successive Refinement

We connect our empirical findings to the classical theory of *successive refinement* [Equitz and Cover, 1991, Rimoldi, 1994, Cover and Thomas, 2006], providing formal conditions for when and why hierarchy-aligned supervision produces steerability.

Setup. Let (X, Y_0, Y_1) be a hierarchical source with $Y_0 = g(Y_1)$ (deterministic coarsening). An encoder produces $\mathbf{z} = [\mathbf{z}_1; \dots; \mathbf{z}_J] \in \mathbb{R}^d$ with prefix $\mathbf{z}_{\leq m} = [\mathbf{z}_1; \dots; \mathbf{z}_m]$. Let $C(d')$ denote the effective capacity (in bits) of a d' -dimensional embedding.

Background: Successive refinement. A source X is *successively refinable* at distortion pair (D_1, D_2) if a two-stage code can achieve the point-to-point rate-distortion optimum at *both* stages simultaneously [Equitz and Cover, 1991]. Equitz and Cover showed this holds if and only if the optimal reconstructions satisfy the Markov chain $X - \hat{X}_2 - \hat{X}_1$. Rimoldi [Rimoldi, 1994] characterised the full achievable rate region: $R_1 \geq I(X; \hat{X}_1)$, $R_1 + R_2 \geq I(X; \hat{X}_1, \hat{X}_2)$. Crucially, No [No, 2019] proved a *universality* result: any discrete memoryless source is successively refinable when the first-stage decoder uses *logarithmic loss*—which is mathematically identical to cross-entropy.

Connecting V5 to successive refinement. The standard successive refinement setup encodes a single source X at two resolutions. Our setup is structurally different: the prefix targets $Y_0 = g(Y_1)$ (a deterministic coarsening) while the full embedding targets Y_1 . We show that this is a *relaxation* of standard SR, not a complication.

Theorem 1 (Hierarchy-Successive-Refinement). *Let (X, Y_0, Y_1) be a hierarchical source with $Y_0 = g(Y_1)$. Assume $C(d/J) \geq H(Y_0)$ and $C(d/J) < H(Y_1)$. Under V5 supervision ($\mathbf{z}_{\leq 1} \rightarrow Y_0$ via cross-entropy, $\mathbf{z} \rightarrow Y_1$ via cross-entropy):*

$$I(\mathbf{z}_{\leq 1}; Y_0) > I(\mathbf{z}_{\leq 1}; Y_1 | Y_0) \quad (\text{coarse-prioritised prefix}) \quad (5)$$

Under MRL ($\mathbf{z}_{\leq 1} \rightarrow Y_1, \mathbf{z} \rightarrow Y_1$): no specialisation.

Proof sketch. The capacity bottleneck $C(d/J) < H(Y_1)$ means the prefix cannot encode all of Y_1 , but $C(d/J) \geq H(Y_0)$ ensures it can encode Y_0 . V5’s prefix cross-entropy loss targeting Y_0 forces the optimal prefix to maximise $I(\mathbf{z}_{\leq 1}; Y_0)$, yielding coarse-prioritised encoding. The formal reduction to standard SR proceeds by decoder mapping. Take any SR code for Y_1 under log loss at both stages (existence guaranteed by No [No, 2019]). Keep the encoder and Stage 2 decoder unchanged. Replace the Stage 1 decoder $q_1(\cdot|m_1) \in \Delta(\mathcal{Y}_1)$ with the coarse marginal $q_0(y_0|m_1) = \sum_{y_1: g(y_1)=y_0} q_1(y_1|m_1)$. The resulting Stage 1 log loss on Y_0 satisfies $H(Y_0|M_1) \leq H(Y_1|M_1)$ (marginalising over a partition cannot increase entropy), so V5’s Stage 1 constraint is *strictly weaker* than standard SR’s. Since Stage 2 is unchanged, V5’s rate region contains the standard SR rate region: any achievable (R_1, R_2) for same-source SR yields an achievable point for V5’s coarse-first regime. MRL distributes capacity across both Y_0 and $Y_1|Y_0$ at every prefix length, destroying the nested structure. \square

The successive refinement framework yields three testable predictions, all confirmed:

Corollary 1 (Sign reversal). *Under inverted supervision ($\mathbf{z}_{\leq 1} \rightarrow L_1, \mathbf{z} \rightarrow L_0$), the bottleneck forces the prefix to encode refinement information, producing $\mathcal{S} < 0$. Confirmed: $\mathcal{S} = -0.018$ (CLINC), -0.025 (TREC).*

Corollary 2 (UHMT collapse). *Under uniform multi-task supervision, no prefix is privileged for either task; the optimiser distributes information uniformly, yielding $\mathcal{S} \approx 0$. Confirmed: $\mathcal{S} = +0.001$ (CLINC), -0.009 (TREC).*

Proposition (Steerability identity). *For any two-stage code (M_1, M_2) over a hierarchical source with $L_0 = g(L_1)$, define $\mathcal{S} := [H(L_0|M_1, M_2) - H(L_0|M_1)] + [H(L_1|M_1) - H(L_1|M_1, M_2)]$. Then $\mathcal{S} = I(M_2; L_1|L_0, M_1) \geq 0$.*

Proof. $\mathcal{S} = -I(M_2; L_0|M_1) + I(M_2; L_1|M_1)$. By the conditional MI chain rule and $H(L_0|L_1, M_1) = 0$ (deterministic coarsening): $I(M_2; L_1|M_1) = I(M_2; L_0|M_1) + I(M_2; L_1|L_0, M_1)$. Substituting gives $\mathcal{S} = I(M_2; L_1|L_0, M_1) \geq 0$. \square

This identity reveals that steerability equals exactly the *refinement information* that Stage 2 provides about L_1 beyond what L_0 and M_1 already encode.

Corollary 3 (Flat supervision converse). *If M_1 is flat-log-loss optimal at rate R_1 (i.e., $H(L_1|M_1) = H(L_1) - R_1$), then $\mathcal{S}_{\text{flat}} \leq \min\{R_2, H(L_1) - R_1\}$.*

Proof. $\mathcal{S} = I(M_2; L_1|L_0, M_1) \leq H(L_1|L_0, M_1) \leq H(L_1|M_1) = H(L_1) - R_1$, where the second inequality is conditioning reducing entropy. Also $\mathcal{S} \leq I(M_2; L_1|M_1) \leq R_2$. \square

As prefix capacity grows ($R_1 \rightarrow H(L_1)$), the flat bound vanishes: $\mathcal{S}_{\text{flat}} \rightarrow 0$. This is an *impossibility* property of the objective— $4.5\times$ more trainable parameters cannot break it.

V5 achievability. Under hierarchy-aligned supervision with $I(M_1; L_1|L_0) = 0$ and $R_1 \geq H(L_0)$, we have $H(L_1|L_0, M_1) = H(L_1|L_0)$. Stage 2 encodes $L_1|L_0$ at rate R_2 ; when $R_2 \geq H(L_1|L_0)$, achievability by standard source coding gives $\mathcal{S}_{V5} = H(L_1|L_0)$.

Separation. When $H(L_0) < R_1 < H(L_1)$ and $R_2 \geq H(L_1|L_0)$: $\mathcal{S}_{V5} - \mathcal{S}_{\text{flat}} \geq H(L_1|L_0) - [H(L_1) - R_1] = R_1 - H(L_0) > 0$. The V5 advantage grows linearly with excess prefix capacity above $H(L_0)$.

Confirmed: MRL with 9.2M parameters yields $\mathcal{S} = +0.003$ on CLINC ($d = 11.6$), $\mathcal{S} = -0.000$ on DBPedia Classes ($d = 7.4$), and $\mathcal{S} = +0.008$ on TREC ($d = 4.4$) vs. V5 head-only at 2.1M parameters.

Theorem 2 (Goldilocks capacity–demand matching, informal). *With fixed $H(Y_1)$ and varying K_0 , steerability peaks at $H^*(L_0) \approx C(d/J)$:*

- $H(L_0) < C(d/J)$: spare capacity leaks $L_1|L_0$ information, reducing \mathcal{S} .
- $H(L_0) > C(d/J)$: by Fano’s inequality, prefix errors degrade coarse classification, reducing \mathcal{S} .

- *Taylor expansion around H^** : $\mathcal{S} \approx \mathcal{S}^* - \alpha(H(L_0) - H^*)^2$, matching the empirical quadratic fit ($R^2 = 0.964$).

Corollary 4 (Product predictor). *At the Goldilocks optimum, $\mathcal{S} \propto H(L_1|L_0) \cdot A_{L_1}^{\text{base}}$. The scaling follows from the steerability decomposition: the coarse specialisation term Δ_0 saturates at the Goldilocks optimum (prefix capacity $\approx H(L_0)$), leaving the refinement gap $\Delta_1 = \text{L1}@j_4 - \text{L1}@j_1$ as the dominant term. Under sufficient total capacity, $\text{L1}@j_4$ scales with the backbone’s ability to extract fine information ($A_{L_1}^{\text{base}}$), while $\text{L1}@j_1$ is suppressed by the prefix bottleneck in proportion to $H(L_1|L_0)$ (more refinement bits are excluded from the prefix). The product $H(L_1|L_0) \cdot A_{L_1}^{\text{base}}$ thus captures both the amount of refinement information available and the model’s capacity to exploit it, explaining the empirical $\rho = 0.90$ across eight datasets.*

Testable predictions. (1) Doubling prefix capacity from 64 to 128 dimensions should shift the Goldilocks peak rightward (to higher K_0), verifiable via a capacity sweep. (2) On naturally deep hierarchies ($H(L_1|L_0) > 5$), the super-linear exponent should produce large absolute steerability—provided the fine task is learnable ($A_{L_1}^{\text{base}}$ not at floor).

10 Related Work

Multi-resolution embeddings. MRL [Kusupati et al., 2022] trains embeddings supporting prefix truncation, but all lengths target the same task. SMEC [Zhang et al., 2025] rethinks MRL training for retrieval compression; Matryoshka Multimodal Models [Cai et al., 2025] apply the nesting principle to visual tokens. Most closely related, Hanley and Durumeric [2025] independently train multilingual Matryoshka embeddings where shorter prefixes capture coarse story themes and longer prefixes capture fine-grained event identity, using progressively stricter similarity thresholds per dimension level. Our work complements theirs by contributing a formal successive-refinement framework explaining *why* prefix–hierarchy alignment works, an explicit steerability metric with four causal ablations isolating alignment as the active ingredient, cross-domain evaluation across eight hierarchical classification datasets and three encoder families, and scaling laws linking steerability magnitude to hierarchy structure.

Dimensional redundancy. Takeshita et al. [2025] show that randomly removing 50% of dimensions causes $<10\%$ performance loss, revealing massive redundancy. We exploit this differently: rather than discarding dimensions for compression, we *structure* them to carry semantically distinct information at each prefix length. Weller et al. [2025] prove that single-vector embedding expressiveness for top- k retrieval is bounded by dimensionality, motivating multi-resolution access patterns.

Hierarchical embeddings. Hyperbolic embeddings [Nickel and Kiela, 2017] represent hierarchies through curved geometry. HEAL [Bhattarai et al., 2025] aligns LLM embeddings with domain hierarchies via contrastive losses and matrix factorisation, but does not offer steerability via dimensional truncation. Concurrent work on coarse-to-fine retrieval [Zhao et al., 2025] and multi-granularity RAG interfaces [Du et al., 2026] address the granularity problem through external pipeline mechanisms (multiple retrievers, separate tools) rather than encoding granularity into the vector itself. Our work provides the first theoretical and causal analysis of *why* prefix–hierarchy alignment produces steerability, grounded in the classical theory of successive refinement.

Sparse and compressed embeddings. CSR [Wen et al., 2025] and CSRv2 [Guo et al., 2026] learn sparse codes as alternatives to MRL, achieving efficiency through selective activation. These address adaptive dimensionality via sparsity; our approach uses hierarchical prefix structure. The two are orthogonal and potentially complementary.

11 Conclusion

We have shown that aligning prefix supervision with semantic hierarchy converts dimensional truncation from a fidelity knob into a *semantic zoom* control—at zero inference cost. The evidence is threefold. *Empirically*, fractal training produces steerability on all eight datasets (pooled $d = 1.49$,

$p = 0.0003$), with magnitude predicted by the product of hierarchy depth and baseline learnability ($\rho = 0.90$). *Causally*, four controlled ablations establish that the specific prefix-to-hierarchy alignment—not hierarchy awareness, not architecture, not optimiser—drives the effect: all conditions are highly significant on CLINC ($d \geq 6.1$), with directionally consistent replication on TREC. A factorial backbone control on three datasets confirms alignment is both necessary and sufficient: $4.5\times$ more trainable parameters without alignment produce near-zero steerability ($d = 11.6$ on CLINC, $d = 7.4$ on DBpedia Classes, $d = 4.4$ on TREC). *Theoretically*, the successive refinement framework explains *why* V5 works (the capacity bottleneck forces coarse-first encoding), *when* it works best (Goldilocks capacity–demand matching, $R^2 = 0.964$), and *why flat supervision cannot*: a formal converse bound proves that steerability under flat log-loss vanishes as prefix capacity grows, while V5’s advantage increases linearly with excess prefix capacity ($S_{V5} - S_{\text{flat}} \geq R_1 - H(L_0)$).

The practical implications are concrete. A single fractal embedding enables query-adaptive retrieval: routing coarse queries to the 64d prefix ($3.7\times$ faster on HNSW) and fine queries to 256d, yielding higher mixed accuracy than MRL at 38% lower compute. A single V5 model replaces a dual-encoder system: its 64d prefix achieves 97.5% coarse accuracy, exceeding a dedicated 256d coarse encoder (95.8%). The synthetic hierarchy experiment provides design guidance: measure $H(L_0)$ and size prefix capacity to match.

The connection to successive refinement suggests this is not an empirical curiosity but a fundamental property of information allocation in hierarchical representations. We release all code, data, and experimental artifacts to support replication and extension.

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A Entropy Allocation Analysis

B Retrieval Benchmark Visualisation

C Three-Level Hierarchy Visualisation

D Workload-Adaptive Pareto Analysis

E FAISS Latency Benchmark

F Full Synthetic Hierarchy Results

See Table 5 for complete results across all 8 coarse partition sizes. The experiment holds total class count fixed at 150 while varying K_0 from 2 to 75 with randomly assigned hierarchies on CLINC-150 text.

Disentangling the Scaling Law: $H(L_0)$ is the True Driver

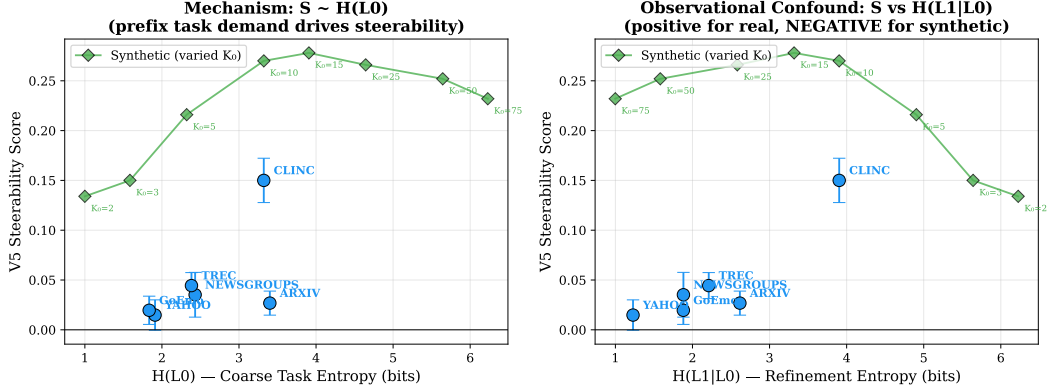


Figure 8: Disentangling the scaling law. **Left:** Steerability vs. $H(L_0)$ (prefix task demand)—the true driver, confirmed by the synthetic experiment. **Right:** Steerability vs. $H(L_1|L_0)$ —a confounded proxy in observational data. Synthetic data (green diamonds) breaks the confound: $H(L_1|L_0)$ anti-correlates with steerability when $H(L_0)$ is varied independently.

Retrieval Benchmark: V5 Enables Coarse-to-Fine Recall via Truncation

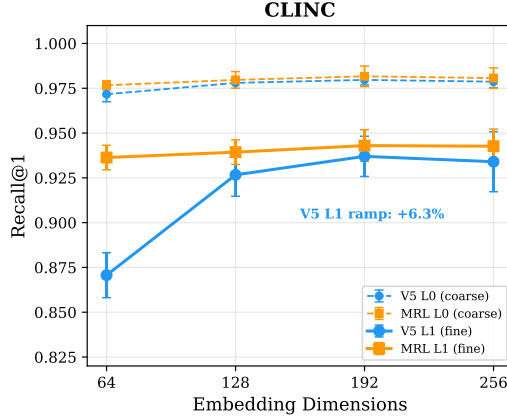


Figure 9: Retrieval benchmark on CLINC-150 (3 seeds). V5 L_1 Recall@1 ramps steeply from 64d to 192d (+6.3pp) while MRL is flat (+0.6pp). Both achieve comparable L_0 Recall@1 ($> 97\%$). The $10\times$ larger ramp demonstrates that prefix specialisation transfers from classification to retrieval.

G Training Convergence

All models converge within 5 epochs of head-only training. Stage 2 (backbone fine-tuning) was tested but provided no improvement, consistent with the frozen backbone providing sufficient representational capacity for hierarchy-aligned supervision.

H Reproducibility

Code and data. All code, trained models, and result JSONs are available at <https://github.com/dl1683/ai-moonshots>. Every experiment uses publicly available datasets loaded via the `datasets` library with deterministic seeded splits.

3-Level Hierarchy: Monotonic Semantic Zoom (CLINC 5-→10-→150)

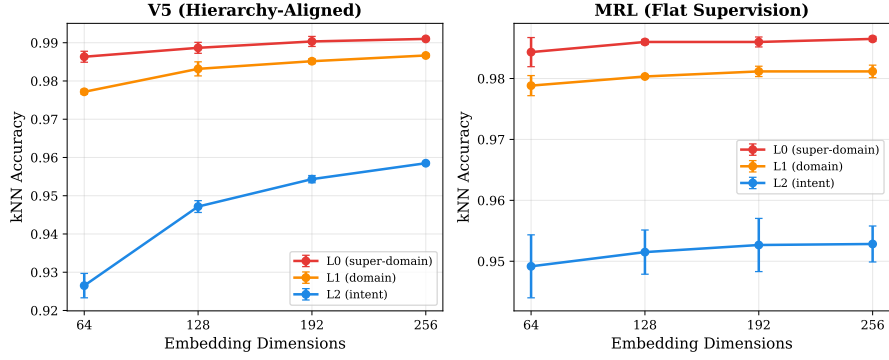


Figure 10: Three-level hierarchy (CLINC, 5→10→150, 3 seeds). **Left:** V5 shows clear level separation— L_2 gains most from additional dimensions while L_0 is near ceiling. **Right:** MRL curves are bunched with minimal ramp at any level.

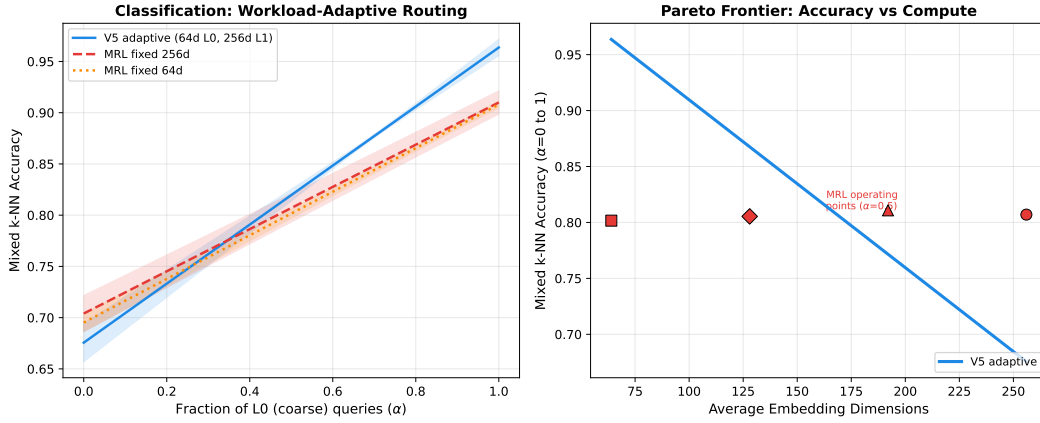


Figure 11: **Left:** Mixed accuracy vs. workload mix α (fraction coarse). V5 adaptive dominates MRL-256d for $\alpha \geq 0.35$. **Right:** Pareto frontier—V5 achieves higher accuracy at lower dimensionality. Bands: ± 1 SD (5 seeds).

Hyperparameters. All experiments use: 5 epochs head-only training, batch size 16, lr 10^{-4} , AdamW with cosine decay, FP16, gradient clipping at 1.0. Prefix sampling [0.4, 0.3, 0.2, 0.1] and block dropout [0.95, 0.9, 0.8, 0.7] are fixed across all datasets and models. No per-dataset tuning.

Compute. Single NVIDIA RTX 5090 Laptop GPU (24GB VRAM). Each training run: ~ 2 min (BGE-small, 33M) or ~ 8 min (Qwen3-0.6B). Full suite: ~ 16 GPU-hours.

I Metric Robustness

We verify conclusions hold under three alternative steerability formulations:

- \mathcal{S}_{AUC} : average steerability over all prefix pairs $k = 2, 3, 4$.
- $\mathcal{S}_{\text{mono}}$: fraction of adjacent pairs with monotonic coarse-decrease / fine-increase.
- \mathcal{S}_{gap} : specialisation gap $(L_0@j_1 - L_1@j_1) - (L_0@j_4 - L_1@j_4)$.

All pairwise rank correlations exceed $\rho = 0.90$ ($p < 0.005$), confirming all four metrics recover the same dataset ordering.

Table 9: FAISS query latency at different dimensionalities (RTX 5090, float32). V5’s 64d prefix queries are $3.7\text{--}5.1\times$ faster than 256d.

Dim	Flat (n=10K)		HNSW (n=100K)	
	Latency (μs)	Speedup	Latency (μs)	Speedup
64d	35	$5.1\times$	39	$3.7\times$
128d	110	$1.6\times$	87	$1.7\times$
192d	146	$1.2\times$	81	$1.8\times$
256d	179	$1.0\times$	145	$1.0\times$

Table 10: Metric robustness: V5–MRL gap under four steerability formulations. Three of four agree V5 > MRL on all 8 datasets.

Dataset	$H(L_1 L_0)$	$\Delta\mathcal{S}_{\text{orig}}$	$\Delta\mathcal{S}_{\text{AUC}}$	$\Delta\mathcal{S}_{\text{mono}}$	$\Delta\mathcal{S}_{\text{gap}}$
Yahoo	1.23	+0.010	+0.013	−0.10	+0.010
GoEmotions	1.88	+0.014	+0.014	−0.07	+0.014
Newsgroups	1.88	+0.035	+0.037	+0.03	+0.035
TREC	2.21	+0.045	+0.039	+0.27	+0.045
arXiv	2.62	+0.028	+0.020	+0.17	+0.028
DBPedia Cl.	3.17	+0.112	+0.106	+0.33	+0.112
CLINC	3.90	+0.143	+0.124	+0.27	+0.143
WOS	5.05	+0.036	+0.027	+0.30	+0.036
V5 > MRL		8/8	8/8	6/8	8/8
Sign test p		0.004	0.004	0.14	0.004

J Per-Seed Steerability Values

Table 11: Per-seed steerability (V5 and MRL) across all eight datasets. Seeds: 42, 123, 456, 789, 1024.

Dataset	V5 \mathcal{S} by seed					MRL \mathcal{S} by seed				
	42	123	456	789	1024	42	123	456	789	1024
Yahoo	+0.016	+0.020	−.004	−.002	+0.044	−.010	+0.016	+0.006	+0.016	−.002
GoEmo	−.002	+0.024	+0.026	+0.006	+0.044	+0.022	+0.022	+0.006	−.012	−.010
News	+0.040	−.012	+0.038	+0.044	+0.066	−.002	+0.022	+0.008	−.018	−.010
TREC	+0.018	+0.062	+0.054	+0.042	+0.046	+0.000	+0.024	−.014	−.002	−.012
arXiv	+0.038	+0.018	+0.012	+0.018	+0.048	−.014	−.010	+0.000	+0.000	+0.020
DBP	+0.118	+0.092	+0.130	+0.130	+0.130	+0.020	+0.008	+0.008	+0.000	+0.002
CLINC	+0.104	+0.178	+0.150	+0.168	+0.150	+0.012	+0.028	+0.006	−.016	+0.004
WOS	+0.032	+0.022	+0.008	+0.052	+0.074	+0.000	−.004	+0.004	−.004	+0.010

K Scaling Trend Robustness

Bootstrap analysis (10,000 resamples): 98.0% of $\rho(H(L_1|L_0))$ values positive. Meta-analysis prediction interval $[-0.70, 3.18]$ reflects heterogeneity ($I^2 = 63\%$) explained by the interaction model.

L Broader Impact

Fractal embeddings add a semantic control knob to existing embedding models without modifying the backbone. The primary application is more efficient retrieval: coarse-first filtering reduces compute by $4\times$ without sacrificing fine-grained accuracy when needed. We do not foresee negative societal impacts beyond those inherent to embedding-based retrieval generally. The method is domain-agnostic and introduces no biases beyond those present in the frozen backbone.

Table 12: Leave-one-out sensitivity for the $H(L_1|L_0)$ scaling trend. WOS (Cook’s $D = 3.68$) is the highest-influence point; the product predictor resolves its deviation without dropping any dataset.

Dropped	k	Spearman ρ	p	Pearson r	p
None (full)	8	0.74	0.035	0.49	0.218
Yahoo	7	0.61	0.144	0.40	0.376
GoEmotions	7	0.64	0.119	0.45	0.318
Newsgroups	7	0.71	0.071	0.47	0.289
TREC	7	0.83	0.021	0.48	0.272
arXiv	7	0.78	0.041	0.50	0.259
DBPedia Classes	7	0.72	0.068	0.49	0.261
CLINC	7	0.72	0.068	0.34	0.457
WOS	7	0.87	0.012	0.91	0.004