Is the Unit Interval “Smaller” than the Unit Square?

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I. Introduction

This article deals with an interesting question about cardinality: Is the unit interval [0, 1] and the unit square [0, 1] × [0, 1] of the same cardinality? That is, does there exist a bijection between [0, 1] and [0, 1] × [0, 1]? Though the answer may be counterintuitive, the answer to this question is affirmative. In the following, we construct a continuous mapping called “space-filling curve,” from [0, 1] onto [0, 1] × [0, 1]. With space-filling curve, we can find one-to-one mappings from [0, 1] × [0, 1] to [0, 1] and from [0, 1] to [0, 1] × [0, 1] respectively. Then by applying Schroeder-Bernstein theorem, we can complete the proof.

Definition 1.1. Equivalence of Cardinality

Let A, B be two sets. If there exists a bijection between A and B, then it is said that A and B are of the same cardinality.

The next theorem provides a sufficient condition to compare the cardinality of two sets.

Theorem 1.2. (Schroder – Bernstein)

Let A, B be two sets. If there is a one-to-one function f: A → B and a one-to-one function g: B → A, then A and B are of the same cardinality.

Proof: We may assume A and B are disjoint. We may arrange this if necessary by replacing A by $\{(a, 0)\colon a \in \mathrm{A}\}$ and B by $\{(b, 1)\colon b \in \mathrm{B}\}$. Let D be the image of f and let C be the image of g. Define a chain to be a sequence of elements of either A or B, that is, a function ϕ: N → (A ∪ B) such that

a) $\phi(1) \in \mathrm{B} \setminus \mathrm{D}$.

b) If for some j we have $\phi(j) \in \mathrm{B}$ then $\phi(j + 1) = \mathrm{g}(\phi(j))$.

c) If for some j we have $\phi(j) \in \mathrm{A}$ then $\phi(j + 1) = f(\phi(j))$.

We see that a chain is a sequence of elements of A $\cup$ B such that the first element is in B $\setminus$ D, the second in A, the third in B, and so on. Obviously each element of B $\setminus$ D occurs as the first element of at least one chain.
Define S $= \{a \in A : a$ is some term of some chain$\}$. It is helpful to note that
S $= \{x \in A : x$ can be written in the form $x = g(f(g(\ldots f(g(y))\ldots)))$for some $y \in$ B $\setminus$ D$\}$.

We set

$$k(x) = \begin{cases} f \quad (x) & if \quad x \in A \setminus S \\ g^{-1}(x) & if \quad x \in S \end{cases}$$

Note that the second half of this definition makes sense because S $\in$ C and because g is one-to-one. Then $k :$ A $\rightarrow$ B. We shall show that in fact k is a bijection.

First note that f and g^{-1} are one-to-one. We show that k is one-to-one as follows: If $f(x_1) = g^{-1}(x_2)$ for some $x_1 \in$ A $\setminus$ S and some $x_2 \in$ S, then $x_2 = g(f(x_1))$. But, the fact that $x_2 \in$ S now implies that $x_1 \in$ S. That is a contradiction. Hence k is one-to-one. It remains to show that k is onto. Fix $b \in$ B. We seek an $x \in$ A such that $k(x) = b$.

Case A : If $g(b) \in$ S, then $k(g(b)) \equiv g^{-1}(\mathrm{g}(b)) = b$. Hence the x that we seek is $g(b)$.

Case B : If $g(b) \notin$ S, then we claim there is an $x \in$ A such that $f(x) = b$. Assume this claim for the moment.

Now the x that we just found must lie in A $\setminus$ S. For if not then x would be in some chain. Then $f(x)$ and $g(f(x)) = g(b)$ would also lie in that chain. Hence $g(b) \in$ S, and that is a contradiction. But $x \in$ A $\setminus$ S tells us that $k(x) = f(x) = b$. That completes the proof that k is onto.

To prove the claim we made in Case B, note that if there is no x $\in$ A with $f(x) = b$ then $b \in$ B $\setminus$ D. Thus some chain would begin at b. So $g(b)$ would be a term of that chain. Hence $g(b) \in$ S and that is a contradiction.

The proof of the Schroeder-Bernstein theorem is complete.

In order to prove the next theorem applied in our proof, we recall axiom of choice.

Axiom of Choice (Zermelo′s Axiom)

Consider a family of arbitrary nonempty disjoint sets E_α indexed by a set A, $\{ E_\alpha : \alpha \in \mathrm{A}\}$. Then there exists a set consisting of exactly one element from each E_α, $\alpha \in$ A.

Theorem 1.3.

Let A, B be two sets. If there exists surjective functions mapping A onto B and from B onto A respectively, then A and B are of the same cardinality.

Proof: Consider two surjective functions ψ and φ map A onto B and B onto A respectively. Let $\varphi^{-1}(a) = Ea$, $a \in$ A, $Ea \subseteq$ B. $\bigcup_{a \in A} Ea =$ B, $E_\beta \bigcap E_\alpha = \phi$ if $\alpha \neq \beta$. By axiom of choice, there exists a subset S_B of B, which consists of exactly one element from each Ea. Therefore, there exists an one-to-one onto mapping $\tilde{\varphi}^{-1}$: A $\rightarrow S_B$. Hence, $\tilde{\varphi}^{-1}$: A $\rightarrow$ B is an one-to-one mapping. In a similar way, we can also construct an one-to-one mapping $\tilde{\psi}^{-1}$: B $\rightarrow$ A. Then by Schroeder-Bernstein theorem, A and B are of the same cardinality.

II. Space – Filling Curve

The space-filling curve we are going to construct is a continuous onto mapping from [0, 1] to [0, 1] $\times$ [0, 1]. We will use Weierstrass M-test to show the continuity of the space-filling function.

Definition 2.1. Uniform Convergence

A sequence of functions $\{f_n\}$ is said to converge uniformly to f on a set S if, for every $\varepsilon > 0$, there exists an N (depending only on ε) such that $n > N$ implies

$$|f_n(x) - f(x)| < \varepsilon, \quad \text{for every } x \text{ in S.}$$

Theorem 2.2. Weierstrass M – test

Let $\{M_n\}$ be a sequence of nonnegative numbers such that $|f_n(x)| \leq M_n$, for n = 1, 2, ... , and for every x in S. Then $\sum f_n(x)$ converges uniformly on S if $\sum M_n$ converges.

Space – filling curve

Let ϕ be defined on the interval [0, 2] by the following formulas:

$$\phi(t) = \begin{cases} 0, & if\ 0 \le t \le \frac{1}{3},\ or\ \frac{5}{3} \le t \le 2, \\ 3t-1, & if\ \frac{1}{3} \le t \le \frac{2}{3}, \\ 1, & if\ \frac{2}{3} \le t \le \frac{4}{3}, \\ -3t+5, & if\ \frac{4}{3} \le t \le \frac{5}{3}, \end{cases}$$

Extend the definition of ϕ to all of R by the equation

$$\phi(t+2) = \phi(t).$$

This makes ϕ periodic with period 2.

Now define two functions f_1 and f_2 by the following equations:

$$f_1 = \sum_{n=1}^{\infty} \frac{\phi(\ 3^{2n-2}t)}{2^n}, \qquad f_2 = \sum_{n=1}^{\infty} \frac{\phi(\ 3^{2n-1}t)}{2^n}.$$

Both series converge absolutely for each real t and they converge uniformly on R. In fact, since $|\phi(t)| \le 1$ for all t, the Weierstrass M-test is applicable with $M_n = 2^{-n}$. Since ϕ is continuous on R, f_1 and f_2 are also continuous on R. Let $f = (f_1, f_2)$ and let Γ denote the image of the unit interval [0, 1] under f. We will show that Γ fills the unit square.

First, it is clear that $0 \le f_1(t) \le 1$ and $0 \le f_2(t) \le 1$ for each $t \in [0,\ 1]$, since $\sum_{n=1}^{\infty} 2^{-n} = 1$. Hence, Γ is a subset of the unit square. Next, we must show that $(a,\ b) \in \Gamma$ whenever $(a,\ b) \in [0,\ 1] \times [0,\ 1]$. For this purpose we write a and b in binary terms. That is, we write

$$a = \sum_{n=1}^{\infty} \frac{a_n}{2^n}, \quad b = \sum_{n=1}^{\infty} \frac{b_n}{2^n},$$

where each a_n and each b_n is either 0 or 1. Now let $c = 2\sum_{n=1}^{\infty} \frac{c_n}{3^n}$ where $c_{2n-1} = a_n$ and $c_{2n} = b_n$, $n = 0,\ 1,\ 2,...$

Clearly, $0 \le c \le 1$ since $2\sum_{n=1}^{\infty} 3^{-n} = 1$. We will show that $f_1(c) = a$ and that $f_2(c) = b$.

If we can prove that

$$\phi(3^k c) = c_{k+1}, \qquad \text{for each } k = 0, 1, 2, ...,$$

then we will have $\phi(3^{2n-2}c) = c_{2n-1} = a_n$ and $\phi(3^{2n-1}c) = c_{2n} = b_n$, and this will give us $f_1(c) = a$ and $f_2(c) = b$. we write

$$3^k c = 2\sum_{n=1}^{k} \frac{c_n}{3^{n-k}} + 2\sum_{n=k+1}^{\infty} \frac{c_n}{3^{n-k}} = (\text{an even integer}) + d_k,$$

where $d_k = 2\sum_{n=1}^{\infty} \frac{c_{n+k}}{3^n}$. Since ϕ has period 2, it follows that

$$\phi(3^k \mathrm{c}) = \phi(d_k).$$

If $c_{k+1} = 0$, then we have $0 \le d_k \le 2\sum_{n=2}^{\infty} 3^{-n} = \frac{1}{3}$, and hence $\phi(d_k) = 0$. Therefore, $\phi(3^k c) = c_{k+1}$ in this case. The only other case to conisder is $c_{k+1} = 1$. But then we get $\frac{2}{3} \le d_k \le 1$ and it follows that $\phi(d_k) = 1$. Therefore, $\phi(3^k c) = c_{k+1}$ in all cases and this proves that $f_1(c) = a$ and $f_2(c) = b$.

Hence, Γ fills the unit square.

It's obvious that there exists a surjective mapping from unit square to unit interval, space-filling curve itself is a surjective mapping from unit interval to unit square. Then by Theorem 1.3, the cardinality of $[0,\ 1] \times [0,\ 1]$ is equal to that of $[0,\ 1]$.

NOTE. Since the unit square and the unit interval are not homeomorphic to each other, the space-filling curve can't be a bijection, i.e., it is not an one-to-one mapping.

III. Disscussion

We will disscuss some extensions in this section. First, we will show that the unit interval $[0,\ 1]$ and $\mathbb{R}^2$ are of the same cardinality.

Theorem 3.1.

$[0,\ 1]$ and $\mathbb{R}^2$ are of the same cardinality.

Proof: First, let $\Gamma : [0,\ 1] \to [0,\ 1] \times [0,\ 1]$ denote the space-filling curve, it's easy to construct a mapping φ from $[0,\ 1] \times [0,\ 1]$ onto $(-\frac{1}{2},\ \frac{1}{2}) \times (-\frac{1}{2},\ \frac{1}{2})$.

$$\varphi(x,\ y) = \begin{cases} (0,\ 0) & if\ \ xy\ =\ 0\ or\ x\ =\ 1\ or\ y\ =\ 1 \\ (x - \frac{1}{2},\ y - \frac{1}{2}) & if\ \ x \in (0,\ 1)\ and\ y \in (0,\ 1) \end{cases}$$

Next, we construct a continuous one-to-one mapping ψ from $(-\frac{1}{2},\ \frac{1}{2}) \times (-\frac{1}{2},\ \frac{1}{2})$ onto $\mathbb{R}^2$.

$$\psi(x,\ y) = (\tan(\pi x), \tan(\pi y)),\ x, y \in (-\tfrac{1}{2},\ \tfrac{1}{2})$$

Therefore, $\psi \circ \varphi \circ \Gamma : [0,1] \to \mathbb{R}^2$ is an onto function.

Let $\sigma : \mathbb{R}^2 \to [0,1]$,

$$\sigma(x,y) = \begin{cases} x & if\ \ x\ \in\ [0,\ 1] \\ 0 & otherwise. \end{cases}$$

It's an onto mapping from $\mathbb{R}^2$ to $[0,1]$.

Hence, by Theorem 1.3, $[0,1]$ and $\mathbb{R}^2$ are of the same cardinality.

We would then provide a theorem for a more general case.

Theorem 3.2.

Let A be a subset of $\mathbb{R}^2$. If there exists an one-to-one mapping σ from $[0,\ 1]$ to S such that S $\subseteq$ A, then $[0,\ 1]$ and A are of the same cardinality.

Proof: By Theorem 3.1, there exists a mapping φ from $[0,\ 1]$ onto $\mathbb{R}^2$. $\because$ A $\subseteq \mathbb{R}^2$, there exists a mapping ϕ from $\mathbb{R}^2$ onto A. Therefore, $\varphi \circ \phi$ is a mapping from $[0,\ 1]$ onto A.

Next, we construct a mapping from A onto $[0,\ 1]$. Consider

$$\psi(x) = \begin{cases} \sigma^{-1}(x) & if\ \ x\ \in\ S \\ 0 & otherwise \end{cases}$$

Then by Theorem 1.3, $[0,\ 1]$ and A are of the same cardinality.

IV. References

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