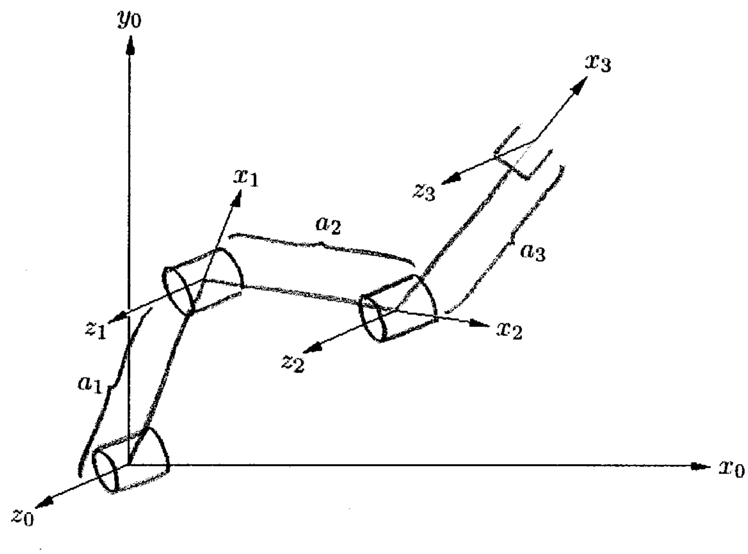
Kinematics of Planar RRR Manipulator



Forward Kinematics

- **Given**: values of the joint variables $\theta_1, \theta_2, \theta_3$
- **Find**: position o_3^0 and orientation R_3^0 of the tool
  That is, find closed-form expressions for the 12 components of the position vector and orientation matrix as a function of the joint variables.
- **Solution**: parameterize the coordinate frames using the DH convention, compute the A matrices, multiply A matrices to get the T matrix, which contains 12 expressions for the tool position and orientation.

$$T_3^0(\theta_1,\theta_2,\theta_3) = \left[\begin{array}{c|c} R_3^0(\theta_1,\theta_2,\theta_3) & o_3^0(\theta_1,\theta_2,\theta_3) \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} c_{123} & -s_{123} & 0 & a_1c_1 + a_2c_{12} + a_3c_{123} \\ s_{123} & c_{123} & 0 & a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Because our manipulator is restricted to the plane, the orientation has a special form.

$$R_3^0(\theta_1,\theta_2,\theta_3) = R_{z,(\theta_1+\theta_2+\theta_3)} = R_{z,\Theta}$$

Consequently, instead of giving the orientation as a matrix with 9 components, we can simply say that the orientation is given by a single value Θ, where

$$\Theta = \theta_1 + \theta_2 + \theta_3 = \text{atan2}(r_{21}, r_{11}).$$

← i.e. given desired orientation in matrix form R_3^0, it is easy to compute Θ

Inverse Kinematics

- **Given**: desired values of position o_3^0 and orientation ($R_3^0 \Leftrightarrow \Theta$) of the tool

$$o_3^0 = \begin{bmatrix} o_x \\ o_y \\ 0 \end{bmatrix}, \; R_3^0 = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Leftrightarrow \Theta = \text{atan2}(r_{21}, r_{11})$$

- **Find**: values of the joint variables $\theta_1, \theta_2, \theta_3$
  That is, find a closed-form expression for the joint variables as functions of the 12 components of the position vector and orientation matrix.

- **Solution**: may not exist, or multiple solutions may exist. Consider two methods of finding solutions (in any exist).

Inverse Kinematics Solution (IKS)

- **Numerical Solution**: compute the forward kinematics solution (as above) and attempt to solve the 12 nonlinear expressions for the 3 unknown joint variables $\theta_1, \theta_2, \theta_3$.

$$\begin{bmatrix} c_{123} & -s_{123} & 0 \\ s_{123} & c_{123} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_1c_1 + a_2c_{12} + a_3c_{123} \\ a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 \end{bmatrix} = \begin{bmatrix} o_x \\ o_y \\ 0 \end{bmatrix}$$

  Since this is a planar case, we can simplify the 9 expressions for orientation into a single expression ($\theta_1 + \theta_2 + \theta_3 = \Theta$) and ignore the third expression for position ($0 = 0$). We have therefore reduced the inverse kinematics problem to that of solving 3 nonlinear expressions for the 3 unknown joint variables $\theta_1, \theta_2, \theta_3$.

$$\left.\begin{aligned} \theta_1 + \theta_2 + \theta_3 &= \Theta = \text{atan2}(r_{21}, r_{11}) \\ a_1c_1 + a_2c_{12} + a_3c_{123} &= o_x \\ a_1s_1 + a_2s_{12} + a_3s_{123} &= o_y \end{aligned}\right\}$$

  given: Θ, o_x, o_y
  Find: $\theta_1, \theta_2, \theta_3$

  Numerical methods may be used to recursively approximate $\theta_1, \theta_2, \theta_3$ using gradient descent or some other algorithm.

  The numerical solution is complicated by the fact that no solution may exist for a given inverse kinematics problem. Moreover, when a solution exists, it is often the case that the solution is not unique. We would like to know: (1) when solutions exist and (2) all possible solutions in closed-form expressions for $\theta_1, \theta_2, \theta_3$.

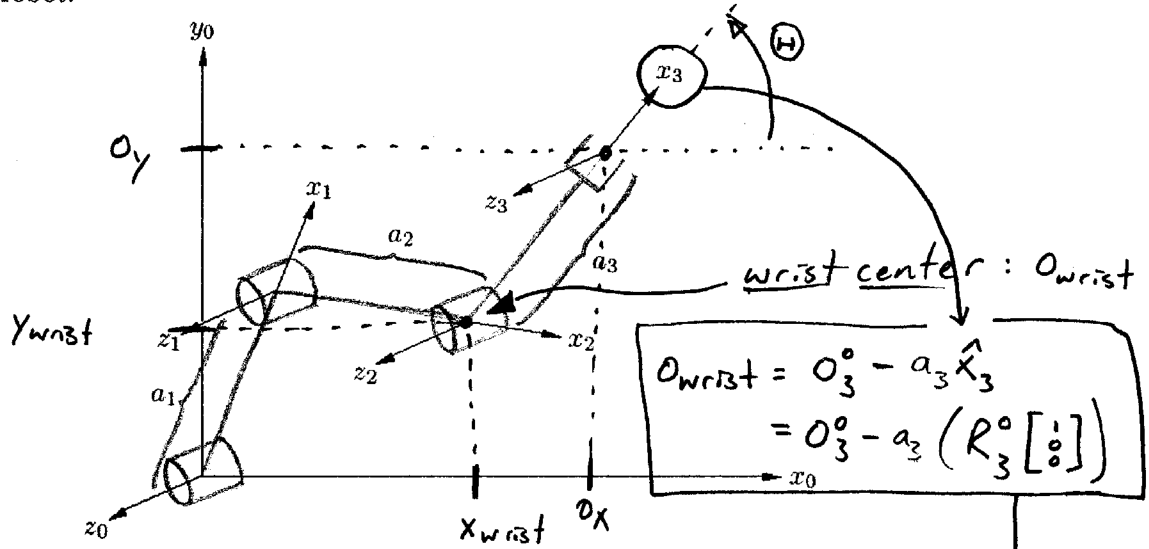
- **Geometric Solution**: when possible, identify a *wrist center* and employ *kinematic decoupling* to separate position and orientation kinematics. Next, use *projection and lifting* to find the joint values that solve the wrist center position kinematics. Finally, use your solution for the wrist center position to find the joint values that solve the overall orientation kinematics.

Geometric IKS Part I: Kinematic Decoupling

1. Identify h, the number of degrees of freedom that determine orientation for this manipulator.

   $h = 1$ because planar.

2. Identify the *wrist center*, a point on the robot whose position is determined by the first $3 - h$ joints of the robot.



3. Given the desired position o_3^0 and orientation ($R_3^0 \Leftrightarrow \Theta$) of the tool, write expressions for the position $o_{wrist} = [x_{wrist}, y_{wrist}, z_{wrist}]^T$ of the wrist center.

$$\boxed{\begin{aligned} x_{wrist} &= O_x - a_3 \cos \Theta \\ y_{wrist} &= O_y - a_3 \sin \Theta \\ z_{wrist} &= 0 \end{aligned}}$$

same expressions

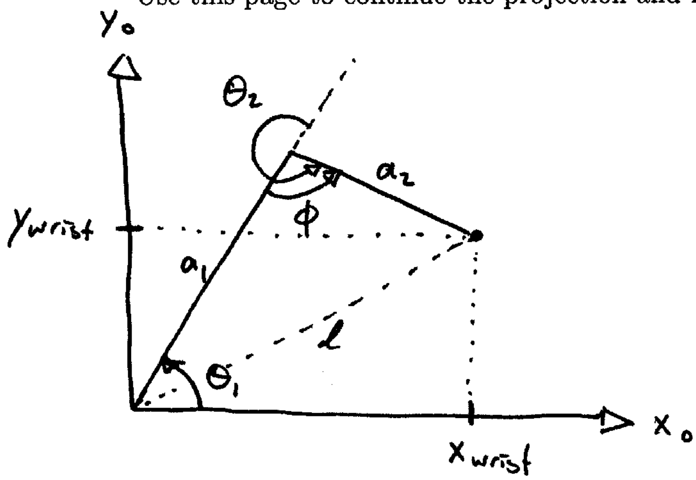
Geometric IKS Part II: Wrist Center *Position*

Given the desired position and orientation of the tool, we have identified where the wrist center must be, i.e. we know from the tool position and orientation the desired position of the wrist center. The task at hand is find closed-form expressions for the joint variables responsible for the position of the wrist center–these are the first $\underbrace{3 - h}_{2}$ joint variables.

1. Sketch the robot from the base frame up to the wrist center.

2. **Projection and lifting**: to solve for a joint variable, first *project* the robot onto a plane perpendicular to the joint axis (if the joint is revolute) or a plane containing the joint axis (if the joint is prismatic). Employ trigonometry to write a closed-form expression for the unknown joint variable. *Lift* that value out of the planar solution and consider it to be a known value for the fully-dimensional robot. Repeat the process of projection and lifting until closed-from expressions are written for all the joints that determine the position of the wrist center.

Use this page to continue the projection and lifting solution of wrist center position.



$$\ell = \sqrt{x_{wrist}^2 + y_{wrist}^2}$$

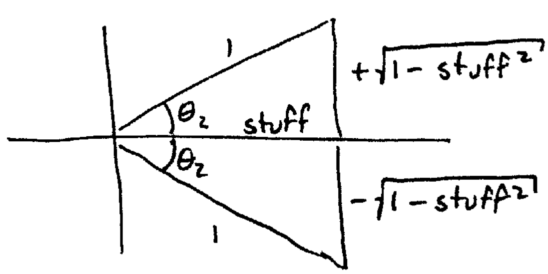
by the law of cosines,

$$\cos\phi = \frac{a_1^2 + a_2^2 - \ell^2}{2a_1 a_2}$$

$$\theta_2 = \phi + \pi$$

$$\therefore \cos\theta_2 = -\cos\phi$$

so $\cos\theta_2 = \dfrac{\ell^2 - a_1^2 - a_2^2}{2a_1 a_2} =: \text{stuff}$

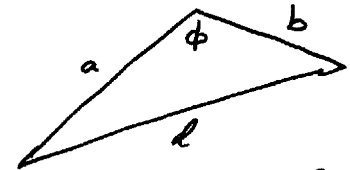


so $\sin\theta_2 = \pm\sqrt{1-\text{stuff}^2}$

$$\frac{\sin\theta_2}{\cos\theta_2} = \tan\theta_2 = \frac{\pm\sqrt{1-\text{stuff}^2}}{\text{stuff}}$$

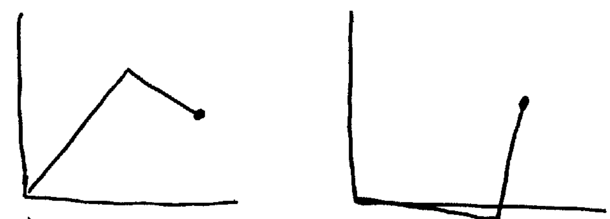
$$\therefore \boxed{\theta_2 = \operatorname{atan2}\left(\text{stuff}, \pm\sqrt{1-\text{stuff}^2}\right)}$$

Aside: Law of Cosines



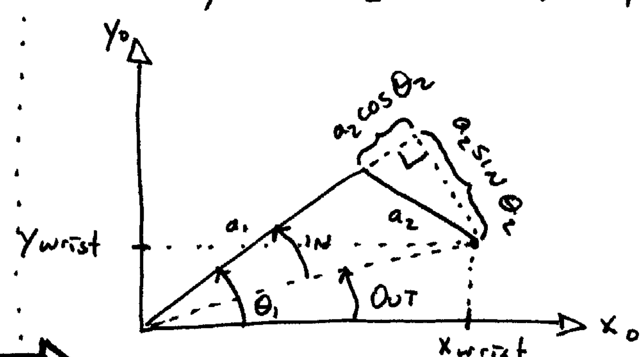
$$\cos\phi = \frac{a^2+b^2-\ell^2}{2ab}$$

In general, $\exists$ two sets of $\{\theta_1, \theta_2\}$ solutions that yield position $(x_{wrist}, y_{wrist})^T$



"elbow-up" "elbow-down"

$\Rightarrow$ Now, use θ_2 to find θ_1.



$IN = \operatorname{atan2}(a_1 + a_2\cos\theta_2, a_2\sin\theta_2)$

$OUT = \operatorname{atan2}(x_{wrist}, y_{wrist})$

$$\boxed{\theta_1 = IN + OUT}$$

we know: $R_3^0 = R_2^0 R_3^2 \implies R_3^2 = (R_2^0)^{-1} R_3^0$

Geometric IKS Part III: Overall *Orientation*

Presently, we know (1) the desired position of the tool o_3^0, (2) the desired orientation of the tool ($R_3^0 \Leftrightarrow \Theta$), (3) the desired position of the wrist center o_{wrist}, and (4) expressions for the first $3-h$ joint variables. To complete the IKS, we must find closed-form expressions for the final $\underset{=1}{h}$ joint variables.

1. Using the known values of the first $3-h$ joint variables, compute the rotation R_{3-h}^0, i.e. the rotation from the base frame to the wrist center.
   Note: this is a forward kinematics problem! We begin with known values for joint variables and we need to compute the rotation. Therefore, the DH convention may be useful here.
   $$R_{3-h}^0 = R_2^0 = R_{z,(\theta_1+\theta_2)} = \begin{bmatrix} \text{known} \\ \text{values !} \end{bmatrix}$$
2. Using the desired orientation of the tool frame w.r.t. the base frame ($R_3^0 \Leftrightarrow \Theta$) and the orientation of the wrist center w.r.t. the base frame R_{3-h}^0, compute the orientation of the tool frame w.r.t. the wrist center R_3^{3-h}.
   $$R_3^{3-h} = R_3^2 = \underset{\text{known}}{(R_2^0)^{-1}} \underset{\text{known (given)}}{R_3^0} = \begin{bmatrix} \text{known} \\ \text{values} \end{bmatrix}$$
3. Write closed-form expressions for the final h joint variables based on the required orientation of the tool frame w.r.t. the wrist center R_3^{3-h}.

$$\text{DH} \implies R_3^2 = R_{z,\theta_3} = \begin{bmatrix} \text{expressions} \\ \text{in } \theta_3 \end{bmatrix}$$

$$\text{above} \implies R_3^2 = (R_2^0)^{-1} R_3^0 = \begin{bmatrix} \text{values} \end{bmatrix}$$

$$\text{set} \begin{bmatrix} \text{expressions} \\ \text{in } \theta_3 \end{bmatrix} = \begin{bmatrix} \text{values} \end{bmatrix}$$

solve the 9 equations in 1 unknown.

In this planar case, $R_2^0 = R_{z,(\theta_1+\theta_2)}$ and $R_3^0 = R_{z,\Theta}$

$$\text{so } \overset{\text{expressions in } \theta_3}{R_3^2 = R_{z,\theta_3}} = \overset{\text{known values}}{R_{z,-(\theta_1+\theta_2)} R_{z,\Theta} = (R_2^0)^{-1} R_3^0}$$

$$\boxed{\theta_3 = \Theta - (\theta_1+\theta_2)}$$

Geometric IKS Part IV: Existence and Uniqueness

We now have closed-form expressions for all joint variables based on the desired position and orientation of the tool. We are prepared to answer the questions of whether solutions of the inverse kinematics problem (1) exist and (2) are unique.

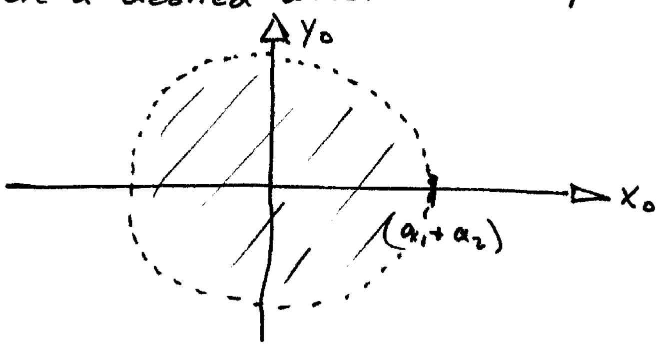
1. Given arbitrary desired tool position and orientation, does a solution to the inverse kinematics problem exist? What restrictions must be placed on the desired position and orientation in order for a solution to exist?

Easiest way to answer this question is to recast it in terms of the wrist center (becomes a question of position alone).

GIVEN: position + orientation

COMPUTE: wrist center position

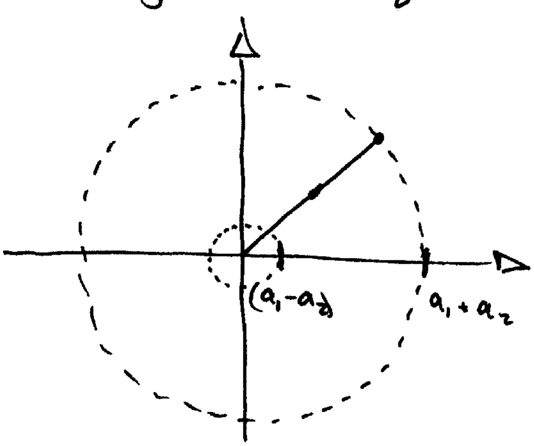
NOW: given a desired wrist center position, does an IKS exist?



for existence of soln, $\|O_{wrist}\| \leq (a_1+a_2)$

2. Given abitrary desired positions and orientations of the tool, how many inverse kinematics solutions exist?

Again, recast question in terms of the wrist center



$\|O_{wrist}\| = (a_1+a_2)$ — 1 soln

$(a_1-a_2) \leq \|O_{wrist}\| < (a_1+a_2)$ — 2 solns

(Special case: $a_1 = a_2$)

$\|O_{wrist}\| = 0$ — ∞ solns