series may be regarded as the expression of an integral : when any series then is proposed, we must endeavour to find the differential expression of which that series is the integral ; and as we can always find the integral of a diffe­rential, at least by approximation, within given limits, we may thence determine, if not the exact, at least the approxi­mate value of any infinite series. We shall now show how this principle may be applied in some particular cases.

Problem I. It is proposed to find the sum of « terms of the series

*x* + 2x2 + 3xs + 4,r4 ∙ ∙ ∙ ∙ -+- *nx".*

Let the sum be denoted by *s.* Then, multiplying all the

terms by —, we have

*1 X*

*sdx*

*— — dx* 4- *2xdx* + 3xstfx ∙ ■ ∙ + *nx"dx.*

*X*

Let the integral of both sides be now taken, and the re­sult is

Z *≈ X* + xi + x3 + *xi ∙ ’ ’ + x^.*

*%' X*

Now the series on the right-hand side of this equation is a geometrical progression, the sum of which is known

X — x” \*l

to be ~~i~~ -- (Algebra, § 56). Therefore

*psdx X —* x" 11

*J X* 1 — *x ’*

and, taking the differentials,

*dx\_dx —* (n + *l)xndx + rιxκ , 1 dx* \_\_\_ (T^xÿ ’

Hence we find

***X — (n* + l)χ"\*l + *nxn \*s***

This result agrees with that formerly found (17) of this article.

Problem II. It is proposed to sum the infinite series

ι 1 I 1 1 i 1 1 . ,

1 3 + 5 7 + 9 n+>\*ic∙

We may consider this series as a particular case of the more

general series

x3 . xs x7 1 „

«—-§- +y — 7^+> &c’

namely, that in which x = 1. Putting therefore the sum *= s,* and taking the differentials, we have

*ds = dx* (1—*x∙ + xt —* x6 + , &c.).

Now the series in the parenthesis is obviously the develop- 1 *dx*

ment of the rational fraction —i—⅛ therefore *els =*

l+a l+χ-

and taking the integral *s* = arc (tan. = x) + *c,* radius be­ing unity. (FluxioNS, § 139.) Now, when x = 0, all the terms of the series vanish, so that in this case «= 0; and a3 when x = 0, arc (tan. = x) = 0; therefore *c,* the constant quantity added to complete the fluent, is 0, and we have simply *s* = arc (tan. = x), and when x = 1, then **s = l** a quadrant = ∙7853982.

Problem III. Required the sura of the infinite series *x , xs* , xa , *xλ ,*

1 ∙ 2 ' 2 ∙ 3 \* 3 ∙ 4 \*" 4 ∙ 5 ’ t

Putting *s* for the sum, and taking the differentials, we get rfx Zx2 x3 x4 x\* ∖

⅛+ 3 + 4~ + ^5 + >ftc∙)

Now the series in the parenthcsis is evidently equal to

—x—Nap. log. (1 —x), (see Algebra, sect, xix.); therefore *ds = -~-~χ* Nap. log. (I —x).

To find the integral, let us put *v* for the function - log.

(I — x), then, taking its fluxion, we have

, rfx , . 1 , *dx*

Λ = -p-×log.(I-x,-7lτ--y

and --J × log. (1 —x) = *dv+i-^-) ;*

therefore, substituting, we get

*j , dx dx*

'x(l—x) x *, dx*

*= dv + Γ=rx>* and taking the integrals,

*s = v —* log. (1 — x) + c

= ⅛⅛=i>-⅛<l-.> + c.

To determine the constant quantity *c,* let us take x = 0; then in this case all the terms of the series vanish, so that s = 0, also log. (1 —x) = log. 1 =0; and since in general log.(l-x)\_l/ .rs x’ ∖ χ

—χ- 2 - 3—’ 4c9 = -1 - 2 -

, &c. when x = 0, then ⅛-i1 r) = — ι ; there-

3 *x*

fore 0 = — I -∣- c, and *c =* I j hence it appears that

\_(l—x)log. (1— x) 1 x ∙^

*Example.* Let x = ⅜, then our formula gives 1 ι 1 . 1 i

l∙2∙2 + 5S∙3∙tf +S∙4∙2s + 4∙5∙8∙ +, &c· - 1 - Nap. log. 2 = ∙3068528.

Problem IV. Let the series to be summed be , »i *m* -U 1 . rn J- 2 .

1 + —x H v-τxi -J 3-,r' 4-, &c.

*n* ‘n-f-l ,n-∣-2 r Putting *s* for this series, let all its terms be multiplied by x “ -1, so that the exponent of *x* in each may be identical with its denominator; the result is

«— = \*∙-> + £ \*" +⅛-"f' + ⅛"t,+>\*c and hence, taking the differentials

χ∙,^^ ,d⅛ -J-(n—l)ix"-¾∕x = (n — l)x"—*adx* -J- *mxn~idx + (m* + 1 ) *xndx* 4- (m -∣- 2) x " h dx-f-, &c.

Let both sides of this equation be now multiplied by *xm ~ ",* and it becomes

x∙"->tir -p (n — *Γ)sχm~adx=(n—l)x,"~adx + mx"l-'dx* + (»» -(- 1 ) *X",dx* + (m + 2) x w, 1dx -∣-, &c.

Putting now the single character *dp* for the fluxional ex­pression which forms the first member of this equation, we get, by taking the fluents of both sides, p=^∈~X"-1 + x™ + x"'1 + x”»2 +, &c.

= ⅛\*"-1 + \*"(l+\* + \*, + \*3+,\*c.)!

but the series in the parenthesis is the development of

—, therefore 1 — x

« — 1 1 x m

*p =* r x ra - 1 -)-1 .

*m —* 1 , 1 — χ