Then MG : GS = sin. MSH : sin. SMG,

and *m :* 1 = sin. SMG : sin. SMH ;

therefore *m ∙* MG : GS = sin. MSH : sin. SMH.

Now S, MSH : S, SMH = MH : HS. Therefore, finally, *m*∙MG : GS = MH : HS.

Now, let MS, the radius of the refracting surface, be called *a*. Let AG, the distance of the focus of incident rays from the surface, be called *r.* And let AH, the focal distance of refracted rays, be called *x.* Lastly, let the sine MX of the semi-aperture be called *e*. Observe, too, that *a*, *r, x,* are to be considered as positive quantities, when AS, AG, AH, lie from the surface in the direction in which the light is supposed to move. If therefore the refracting surface be concave, that is, having the centre on that side from which the light comes ; or if the incident rays are di­vergent, or the refracted rays are divergent ; then *a, r, x,* are negative quantities.

It is plain that HS = *x — a;* GS = *r* — *a;* also AX

= nearly. HX = x—GX=r— ∏-∙ Now add 2α *j 2a 2a*

to HX and to GH their differences from MH and MG,

e2 e2

which (by the Lemma) are —and— ∙ We get MH = *x*

c^ e~ C"

— ir + —, and MG = *r — —* -4- -r-. In order to shorten

2α *2x 2a* 1 2r

our notation, make *k = -— -.* This will make MG = *r a r*

2 '

Now substitute these values in the analogy MH : HS

e, e3

*= m* ∙ MG : GS ; it becomes *x—* -—U — : *x — a =. mr* 2α , *2x*

— ^^2~^ ' : *r — a* (or *ark*), because *k = r—a/ar,* and *ark = r — a.* Now multiply the extreme and mean terms of this analogy. It is evident that it must give us an equation which will give us a value of *x* or AH, the quantity sought.

But this equation is quadratic. We may avoid the so­lution by an approximation which is sufficiently accurate, by substituting for *x* in the fraction — (which is very small in all cases of optical instruments) an approximate value very easily obtained, and very near the truth. This is the focal distance of an infinitely slender pencil of rays converging to G. This we know by the common optical theorem to *be,* β”ίΓ ■-—. Let this be called *φ* ; if we substi-

(m—l)r→=α

tute *k* in place of - — -, this value of ® becomes ≡ - ,.

*t ar m — ah*

This gives us, by the by, an easily-remembered expression (and beautifully simple) of the refracted focus of an infi­nitely slender pencil, corresponding to any distance *r* of the radiant point. For since *φ* = *am/m-ak*; - must be

*m — ak m ak* 1 *k ...*

= = = . We may even ex­am *am am a m*

press it more simply, by expanding *k,* and it becomes 1 \_ i\_2 £

*φ - a ma mr*

Now put this value of — in place of — in the analogy em­ployed above. The first term of the analogy becomes

et e2 Ae5 *kei '*

*X~ 2:i + 2^a ~* 2i, °r *X ~ 2m.* The analogy now becomes

*mhe'"x — — : x — a≈ mr* τr- : *arh.* Hence we ob-

*2m 2*

*...... .∙ rnke,x mhae-*

taιn the linear equation *mrx —— — mra -∣ -—*

*QTh- e~*

*= arhx* ; from which we finally deduce

*. , . arK2ei*

*mra — !; make2 -*

*- 2ηι mr — arh —* 2

We may simplify this greatly by attending to the ele-

*x* I f7 r mentary theorem in fluxions, that the fraction —- ■ ,- dif-

*y + dg*

fers from the fraction by the quantity ∙^^^ x-^ ; this being the fluxion of-. Therefore^-^^^ ≡ - 4- --c^,

*y y + dg y y*

Now the preceding formula is nearly in this situation. It may be written thus : *mra— (∖rnake-* -J- -r^

' ■ Here the last terms of the nu- *mr — arh — rnhet*

numerator and denominator are very small in comparison with the first, and may be considered as the *dx* and *dy,* while *mra* is the *x,* and *mr — ark* is they. Treating it in this way, it may be stated thus :

*ma*

x — ∕ ^f^

*m — αh*

*(mra) mh — (mr — arh) (mhα* -L

∖ ”» ×. !J r\* *(tn — ah)2 2*

The first term *ma/m-ak*is evidently = *φ,* the focal dis­tance of an infinitely slender pencil. Therefore the aber­ration is expressed by the second term, which we must en­deavour to simplify.

Now the numerator of the small fraction, by multiplica­tion, may be expressed thus :

„ , ∕r⅛2 *rthi r3kz∖ , ,* m2α\*( + ) les.

*∖ rn rriiα ‘ m ∙i ∕ \**

The denominator was *r2(m — ak)2,* and the fraction now *miat f k∙ , ⅛3∖, . ,. , ∙ ..*

becomes ; r3 ( -∣ )⅛es, which is evιdent-

*(m — αh)2 ∖mr rnla 1 m∙'J 2*

*(ht kt k3∖ ei*

— + Now recollect that *h — mr rn-a* , rn3/ 2

Therefore⅞st¾l-l) = ⅛-4∙ There- *a r nr nV ∖a rj nV a miτ*

⅛2 jp

fore, instead of— —-f write —i —, and we get the

*ma m nι τ*

*e . . ( h3 h3 h2 h2* ∖ e2 *∕ k3 mh3*

*∖ m3 rni m2r mr I* 2 r ∖m3 m3

*rnk2 mik3∖e2 .* , „ 1—*f,, mhf∖ei*

— —r -4 r- ) —i, which is equal to β2 *r- ( h3 — —* 1-,

*m3r ' m3rI2 Ί γ m2 ∖* r /2

and finally to — *f (k3 — ”-^) ↑-.*

Therefore the focal distance of refracted rays is rn — 1 ∕ *mh2∖ e3 x=^-^~^-{k--)2∙*

This consists of two parts. The first, *φ,* is the focal dis-