been found to be the actual difference of two successive semivibrations. The accuracy of this result, however, must depend on the mutual compensation of its errors ; for the approximation supposes, that if the resistance vanished at the lowest point, the subsequent retardation would be such as to diminish the space by the effect of the diminu­tion of the velocity acting uniformly through the remainder of the vibration, while in fact the diminution of the space from this cause would be simply equal to a part of the arc proportional to the diminution of the velocity, since the arc of ascent is simply as the velocity at the lowest point. Hence it is obvious, that the effects of the resistance are too much complicated with the progress of the vibration to allow us to calculate them separately ; and accordingly, when the resistance is as the square of the velocity, or as sin.2 *x,* the diminution of velocity is expressed by 1/2*x* - 1/2 sin. cos. *x,* and that of the space by 1/4*x*2 *—* 1/4sin.2 *x,* which, at the end of a vibration, becomes 1/4*π*2 instead of *π* ; that is, since the distance of the point of greatest velocity is here *ς* = *Dλ2,* 1/4*π*2Dλ2 = 2∙467 Dλ2, while the more accu­rate mode of computation has shown that the true diminu­tion of the space is 2∙667 *D*λ2. (Theorem G.) If we chose to pursue the mode of approximation here suggest­ed, with accuracy, it would be necessary to consider the resistance as a periodical force acting on a pendulum ca­pable of a synchronous vibration, as hereafter in Theorem K, Schol. 1.

Theorem J. \*\*\*\*\*\* If ⅛ + *Bs* 4- *M* sin. *Ft =* 0, we may β

satisfy the equation by taking *s =* sin. *(VB.t)* 4- ***~~ρρ\_ρ~~*** sin. *Ft.*

d.^

*Demonstration.* The value of *s* here assigned gives us

*MF* ddi

= √jB cos. *VBt +* ***~~pp \_ β~~ ~~cos~~~~∙~~ ~~Fl~~~~>~~ ~~and~~ ~~= — -~~~~βsin~~~~∙~~*** √2⅛— sin. *Pl .* so that l-i⅛ = 2?sin. √*Bt*

*PF — B* di2

*MB . „ . .d, MFF . „*

4- ***~~pp\_~~****-ß sιn∙ Pl — β sιn∙ — FP~ΣΣ'B sιn∙ f, = MB —MFF . e,j „ .*

—„„ ■—5- sin. *Ft = — M* sin. *Ft.*

*II — ιi*

*Corollary* 1. If, in order to generalize this solution, we make *s = α* sin. *√Bt* + *β* cos. *√Bt* + *γ* sin. *Ft* + *ε* cos. *Ft,* we may take any quantities at pleasure for α and *β*, according to the conditions of the particular case to be in­vestigated ; but *ε* must be = 0 ; that is, the motion will always be compounded of two vibrations, the one depen­dent on the length of the pendulum, or on the time requir­ed for the free vibration, indicated by √*Bt,* the other syn­chronous with *Ft,* the period of the force denoted by *M ;* the latter only being limited to the condition of beginning and ending with the periodical force.

*corollary* 2. In the same manner, it may be shown that the addition of any number of separate periodical forces, indicated by the terms *M*' sin. *Ft, M"* sin. *F''t, ..* ., will add to the solution the quantities *M'/F'F'-B* sin. *M''/F''F''-B* sin. *F''t,* and so forth.

*Example* 1. Supposing a pendulum to be suspended on a vibrating centre, and to pass the vertical line at the same moment with the centre, we may make α and *β* = 0, and *s* = *M/FF-B* sin. *Ft* only; the vibration being either di­rect or reversed, according as *F* is less or greater than

√*B*, or than √32/*l*, which determines the spontaneous vi­bration of the pendulum.

*Example* 2. But if the ball of the pendulum be supposed to begin its motion at the moment that the centre of suspension passes the vertical line, we must make *s*= *M/FF-B* (sin. *Ft —* cos. √*Bt),* and the subsequent motion of the pendulum will then be represented by the sum of the sines of two unequal arcs in the same circle ; and if these arcs are commensurate with each other, the vibration will ulti­mately acquire a double extent, and nearly disappear in a continued succession of periods, provided that no resistance interfere. And the consequences of any other initial con­ditions may be investigated in a manner nearly similar. Thus, if the time of free vibration, under these circumstances, were 1/3 of the periodical time, the free vibration, in which the motion must be supposed initially retrograde, in order to represent a state of rest by its combination with the fixed vibration, would have arrived at its greatest excur­sion forwards, after three semivibrations, at the same mo­ment with the fixed vibration, and after three complete vibrations more would be at its greatest distance in the op­posite direction, so as to increase every subsequent vibra­tion equally on each side, and permanently to combine the whole extent of the separate arcs of vibration. But in this and in every other similar vibration, beginning from a state of rest in the vertical line, that is, at the point where the periodical force is evanescent, the effect of the free or sub­ordinate vibration with respect to the place of the body will obviously disappear whenever an entire number of ιsemi- vibrations has been performed.

*corollary* 3. The paradox stated in the fourth scholium on the last theorem may be illustrated by means of this pro­position, and will serve in its turn to justify the mode of computation here employed in a remarkable manner. It has been observed in Nicholson’s Journal for July 1813, that the mode of investigating the effects of variable forces, by resolving them into parts represented by the sines of multiple arcs, and considering the vibrations derived from each term as independent in their progress, but united in their effects, may be applied to the problem of a pendulum vibrating with a resistance proportional to the square of the velocity ; and that for this purpose the square of the sine may be represented by the series sin.2 *x*= ∙8484 sin. *x* — ∙1696 sin. 3*x*— ∙0244 sin. *5x—* ∙00813 sin. *7x —* ∙0029 sin. 9x—∙0013 sin. 11*x—...* Now, if we employ this

series for resolving the resistance supposed in Theorem G into a number of independent forces, the greatest resistance being measured by *A*/√*Bλ*, we shall have ·8484 *A*/√*Bλ* for the part supposed to be simply proportional to the velocity, whence, from Theorem H, we have ∙8484*πA*/√*Bλ* for the corresponding diminution of the vibration ; that is,

2∙6653*A*/√*Bλ*. But it has been observed, in the preceding corollary, that the *place* of the pendulum will not be at all affected by any subordinate vibration after any entire num­ber of complete semivibrations ; and the slight effect of the *velocity* left in consequence of these subordinate vibrations may here be safely neglected, so that 2∙6653 *A*/√*Bλ* may be considered as the whole effect of the resistance with re­spect to the space described, which differs only by 1/2000 of