The method now explained is that which has been fol­lowed in computing all the triangles of the Ordnance sur­vey. The following, which is generally adopted in the con­tinental measurements of geodetical arcs, is of somewhat easier application.

\*\*\*As before, let *a* -÷- r, *b* -÷- r, *c* -÷- r, be represented re­spectively by α, *β, γ.* We have then

\_ . cos. *a —* cos. *β* cos. *γ*

Cos. A = 7—⅛-7 — ;

sm. p sin. *γ* and since the arcs *a, b, c,* are small in comparison of *r*, a sufficient approximation will be obtained, if, in the develop­ment of the sines and cosines in terms of the arc, we retain the terms only which involve powers not higher than the fourth, that is, if we make

\*\*\*r — ι α8 α4 . a \_ a β3

Cos,α = l--2 + *^^.β = β-τ-3i* and also make a like substitution for cos. *β,* cos. *γ,* and sin. *γ∙* With these values, the above equation becomes Cos A = ⅛(3i + ∕-αi) + ⅛(≈,-ffl-∕)-⅜ffl∕ l⅜ (l-⅜(<? + /))

- Ê+ ^∕—flt\* . <χ\* + ffl + 7\*-2α,3--2>87t-23g78

*- 2βγ* 24∕3y

Restoring the values of *a, β, γ,* and denoting the numera­tors of the two terms by M and N respectively, we have cm∙a=⅛+2⅛ -W

In a plane triangle whose sides are respectively equal in length to the three arcs α, *b, c,* let A' be the angle opposite to the side α ; then

oa∙=¾z≤="

On squaring both sides of this equation, and substituting 1 —. sin., A' for cos.! A', we get — *W ci* sin? A' = α4 + *bi* + c4 — 2a2 *P — 2ai* c2 — 25s *ci* = N. In consequence of these two values of M and N, the equation (8) becomes 0e

Cos. A = cos. A' — —5 sin.5 A'.

6ra

If we now assume A=A'-∣- *x,* then a: will obviously be a very small arc, and we may suppose cos. *x* = 1, and sin. *x = x,* without sensible error. Hence cos. A = cos. A' — *x* sin. A', 6c

and we have therefore *x≈7r-3* sin. A, ; whence

6rs

A = A' + Ä sin∙ a'∙

6r5

Now, ½ *bc* sin. A' is the area of the rectilineal triangle whose sides arc *a*, *b*, *c*, and which docs not sensibly differ from the spherical triangle ; therefore, denoting this area by S, and observing that a similar result will be found for B and C, we shall have \*\*\* A = A' + ∣ (S -j- rs), B = B'+ ⅜(S÷ r2), C = C + J ( S -j- r2) ; therefore, since Λ' -f- B' + C' = 180°, A + B + C — (8 ÷ rs) = 180°, so that S -÷- r2 may be regarded as the excess of the three angles A, B, C of the spherical triangle above two right angles.

From this demonstration it follows, that if from each of the angles of any small triangle on the surface of a sphere or spheroid (for the proposition holds true for both surfaces) one third of the spherical excess be deducted, the sines of the angles thus diminished will be proportional to the lengths of the opposite sides, and consequently the sides may be computed as if the triangle were rectilineal. It is to be remarked, that the angles to be diminished by one third of the spherical excess, are not the observed angles but the horizontal angles corrected for the errors of observation.

As an example of this method, we shall take the same triangle as before.

The three angles having been observed, the first step is

to compute the spherical excess E from the formula (1), and for this purpose we take the same value of the radius of the oblique circle as was determined above, namely, log. *r* = 7∙32112. With this radius, the constant part of the spherical excess, namely, 648000÷2r2 *π*, becomes 0∙37116, and hence the formula (1) becomes

Log. E = log. *a* + log. *b* + log. sin. C + 0∙37116. Approximate values of *a* and *b* (one of them may be the given side) must now be found. Now, we found above log. *a* = 5∙63039, log. *b* = 5∙70111 ; and we have also log. sin. C = 9∙83842. Adding these and the constant number 0∙37116 into one sum, we get log. E = 1·54108, and con­sequently E = 34'∙760. This value of the spherical excess agrees exactly (as it ought) with the sum of the three chord corrections found above.

Having computed the spherical excess, the error of the three observed angles is found, and apportioned among the angles in the reciprocal proportion of their respective weights, and the correct spherical angles deduced in the same manner as already explained in describing the other method. Each of the spherical angles is then diminished by 1/3 E = 11"∙582/3; and the results are the *mean* angles from which the sides are to be calculated. They are as under :

|  |  |  |
| --- | --- | --- |
| **Stations.** | **Corrected Spherical Angles.** | **Mean Angles for Calculation.** |
| Benlomond, A... Cairnsmuir, B... Knocklayd, C....  Sum | 56° 43' 28''·62  79 42 29 ·15  43 34 36 ∙99 | 56° 43' 17''∙04  79 42 17 ∙56  43 34 25∙40 |
| 180 0 34 ∙76 | 180 0 0 ∙00 |

With these mean angles, and the given side *c* = 352,037·62 feet (in this case the arc is taken, and not the chord), we can now compute the other sides *a, b.*

\*\*\*Log. *c* (= 352,037∙62) = 5∙5465S9l

Log. sin. A (56’43’ l7^∙0f) = 9∙9222l27 Ix>g. sin. B (79 42 17 ∙56) = 9∙9929511 Co. Iog.sin. C(43 34 25 ∙40) = 04615997

Log. *a* = 5∙63040l5 Log. *b* =5∙70 II399 whence *a* = 426,974∙06 feet, and *b =* 502,504∙42 feet. These results agree almost exactly with those obtained by the reduction to the chord angles, the greatest difference being less than a tenth of a foot in a distance exceeding ninety-five miles.

On comparing the two methods of calculation which have now been explained, it will be obvious that the latter, or that of Legendre, is the simpler of the two, inasmuch as it requires only the calculation of the spherical excess by a single operation, whereas the reduction to the chords re­quires an equivalent operation for each of the three angles. The calculation of the chord angles is however, in prac­tice, rendered easy by an auxiliary table, in which the cor­rection is given for different values of the angle, and of the sum and difference of its containing sides. But the sphe­rical excess may be entered in a table in the same manner. A sufficient approximation to this quantity, or rather to the area of the triangle on which the excess depends, may frequently be obtained by making a plan of the triangle, and measuring the length of the perpendicular upon the known side by means of a scale of equal parts. In fact, if the three angles have been equally well observed, so that the error is to be divided equally among them, the calcula­tion of the spherical excess is not even necessary for com­puting the sides; as the same mean angles will be obtained by diminishing each of the observed angles by one third of the excess of the sum above 180°. Tables for the reduc­tion of the spherical angles to the plane of the chords, and