of the sphere made by two great circles at their intersec­tion : it is identical with the angle made by two straight lines in the planes of the circles which touch them at their intersection ; and it is also the same as the angle made by the planes of the circles.

Let AEB, AFB, be two great circles that intersect each other at A and B (Fig. 2). From O the centre of the sphere, and A the intersection of the circles, draw OE and AP in the plane of the circle AEB, and OF, AQ in the plane of the circle AFB perpendicular to AB, the common section of the circles ; the straight lines AP, AQ, will touch the circles at A, and form an angle PAQ, which is the spherical angle at A; and since PAQ is equal to EOF (Geometry, Part ii. 10, 1), the inclination of the planes of the circles, this last also expresses the spherical angle at A.

Cor. 1. the adjacent spherical angles which one great circle makes with another, are together equal to two right angles ; and the vertical or opposite angles made by two great circles which cut one another are equal.

Cor. 2. The spherical angle made by two great circles is measured by an arc of a great circle whose pole is at their intersection.

Def. VII. A spherical triangle is a figure on the surface of the sphere comprehended by three arcs of great circles, each of which is less than a semicircle.

If planes be supposed to pass along the arcs AB, BC, AC, which are the sides of the triangle ; these will pass through O, the centre of the sphere, and form at that point a solid angle O (Geometry, Def. vi. of Part ii).

Τηεοrεμ III. Any two sides of a

spherical triangle are together greater than the third ; and the sum of the arcs which are its sides is less than the circumference of a great circle.

The arcs AB, AC, BC, arc the measures of the plane angles AOB, AOC, BOC, about the solid angle O. Now, any two of these plane angles are greater than the third (Geometry, Part ii. 16, 1), therefore any two sides of the spherical triangle ABC is greater than the third : and because the sum of the plane angles about the solid angle is less than four right angles (17) ; therefore the sum of the sides of the spherical triangle, which are their measures, must be less than the sum of four quadrants, that is, less than the circumference of a great circle.

Theorem IV. If about the angular points of a spherical triangle, as poles, there be described three great circles on the sphere; these by their intersections will form a triangle which is said to be *supplemental* to the former; and the two triangles are such, that the sides of the one are the supplements of the arcs which measure the angles of the other.

Let ABC be a spheri­cal triangle; let EF be an arc of a great circle whose pole is A, and FD an arc of a great circle whose pole is B, and DE an arc of a great circle whose pole is C ; these three arcs forming the triangle DEF. Let DB be an arc of a great circle passing through the points D, B, and DC an arc of a great circle passing through the points D, C. Because B is

the pole of the arc DF, the arc BD is a quadrant (Th. II.), and because C is the pole of the arc DE, the arc CD is a quadrant. Since then the arcs BD, CD, are both quadrants, D is the pole of a great circle that passes through the points B, C ; but only one great circle can pass through two points on the sphere not directly opposite, therefore D must be the pole of the arc BC. In the same way, it will appear that E is the pole of the arc AC, and F the pole of the arc AB. Produce the arcs AB, AC, if necessary, till they meet the arc EF in G and H ; the point A being the pole of the arc GH, the angle A will be measured by GH. Now EH and FG being both quadrants, EH + FG = 180° ; but EH + FG = EF + GH, therefore EF + GH = 180°, that is, EF, a side of the triangle DEF, is the supplement of GH, the measure of the angle A. In the very same way, it will appear that DF is the supplement of the angle B, and DE the supplement of the angle C ; and because the triangle ABC may be formed from the triangle DEF exactly in the same way that DEF is constructed from ABC, namely, by describing great circles about the angles D, E, F, as poles, the triangles and their relations to each other may be interchanged. Hence it appears that BC, AC, AB, the sides of the triangle ABC, are the supple­ments of D, E, F, the angles of the triangle DEF.

Theorem V. The three angles of any spherical triangle arc greater than two right angles, but less than six.

In the triangle ABC (fig. 4), the measures of the angles A, B, C, together with DE, EF, DF, the sides of the sup­plemental triangle DEF, make six right angles. (Τη. IV.) Now the sum of the sides of the triangle DEF is less than four right angles (Tn. III.), therefore the sum of the three angles A, B, C, is greater than two right angles ; and be­cause the three inward angles of any triangle, together with the adjacent exterior angles, are equal to six right angles, therefore the inward angles alone are less than six right angles.

4. We might proceed in this way, establishing the whole doctrine of spherical trigonometry by a series of synthe­tical demonstrations ; such was the practice of the ear­lier writers on this subject. This however has now given way to a more convenient and elegant method of proceed­ing, which dispenses with complicated diagrams represent­ing on a plane surface circles on a sphere. It deduces by the angular calculus the whole theory of spherical trigo­nometry (including also the theorems already here demon­strated) from a single proposition, in a manner probably first followed out by Bertrand of Geneva in his *Développe­ment nouveau de lu Partie Elémentaire des Mathématiques* (Geneva, 1778). Lagrange seems to have overlooked this work, for in his elegant memoir, entitled *Solutiοns de quelques Problèmes relatifs aux Triangles Sphériques. avec une Ana­lyse complète de ces Triangles,* given in vol. ii. of *Journal de l’Ecole Polytechnique,* he has attributed the theorem to De Gua, who also gave it in the Memoirs of the Academy of Sciences for 1783.

5. Our article on the Arithmetic of Sines (Algebra, sect 25) is an example of what may be done by the power of anaiysis alone in deducing an extensive theory from a few simple geometrical principles. Such is the way in which Lagrange and later writers treat of spherical trigonometry. We shall also follow the method of Delambre, who, in his *Astronomie Théorique et Pratique,* chap. x., instead of nu­merous rules deduced from geometrical constructions, for determining when angles or arcs are less or greater than quadrants, considers their sines and cosines as positive or negative quantities, according to the principles of analytic geometry, as explained at article 225 in Algebra. From what has been there explained, it appears that,

(1.) A sine is *positive* when the arc is less than 180°, but *negative* when the arc i« between 180° and 360°.