term 1/6(n3 + 5n + 6), where for the same values of *n* the series

will be 1, 2, 4, 8, 15, 26, .. The series may contain nega­tive terms, and in forming the sum each term is of course to be taken with the proper sign.

2. But we may have a given law, such as either of those just mentioned, and the question then arises, to find the sum of an indefinite number of terms, or say of *n* terms (n standing for any positive integer number at pleasure) of the series. The expression for the sum cannot in this case be obtained by actual addition ; the formation by addition of the sum of two terms, of three terms, &c., will, it may be, suggest (but it cannot do more than suggest) the expression for the sum of *n* terms of the series. For instance, for the series of odd numbers l + 3 + 5 + 7 + ..., we have 1 = 1, 1 + 3 = 4, 1 + 3 + 5 = 9, etc. These results at once suggest the law, 1 + 3 + 5 ... + (2n - 1) = n2, which is in fact the true expression for the sum of *n* terms of the series ; and this general expression, once obtained, can afterwards be verified.

3. We have here the theory of finite series : the general problem is, *un* being a given function of the positive integer n, to determine as a function of *n* the sum u0 + u1 + u2 ... + un or in order to have *n* instead of *n* + 1 terms, say the sum u0 + *u1 + u2 . . + un-* 1*.*

Simple cases are the three which follow.

(i.) The arithmetic series,

*a + (a + b)* + (*a* + 2b).. + (*a* + *n —* 1)*b* ;

writing here the terms in the reverse order, it at once appears that twice the sum is = *2a + n* - 1*b* taken *n* times : that is, the sum = *na* + 1/2(*n* - 1 )*b*. In particular we have an expression for the sum of the natural numbers

1 + 2 + 3... + *n* = 1/2 *n(n* + 1),

and an expression for the sum of the odd numbers 1 + 3 + 5 .. + (2*n* -1) = *n2.*

(ii.) The geometric series,

*a + ar + ar3... + arn-1 ;*

here the difference between the sum and *r* times the sum is at once seen to be = *a — arn,* and the sum is thus

1 - *rn*.

= Uj \_ ~ ; in particular the sum of the series

l+r + r2.. + rn-1 = ^-.

1 - *r*

(iii.) But the harmonic series,

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*a a + b* α + 20'" α + (τι-l)0,

1,1.1 1 1 , . , or say ι÷2^b3∙ · *' + n,* i\* does not admit of summation;

there is no algebraical function of *n* which is equal to the sum of the series.

4. If the general term be a given function un, and we can find vn a function of *n* such that v,,+1 - *vn = un,* then we have *u*0 *= v*1 - v0, *u*1 *= v2 - v*1*, u*1 *= v3~v9..un* = ‰+ι - vn ; and hence τ∕θ + *ul + u2 .. + un* = vn+ι - ⅛0,—an expression for the required sum. This is in fact an application of the Calculus of Finite Differences. In the notation of this calculus vn+l - *vn* is written ∆vn ; and the general inverse problem, or problem of integration, is from the equation of differences ∆vn = *un* (where *υn* is a given func­tion of *n)* to find *vn.* The general solution contains an arbitrary constant, *vn = Vn + C* ; but this disappears in the difference vn + 1 - vθ, As an example consider the series

*uo+ u1... + un=*0 +1 + 3 .. + 1/2n(n +1) ; here, observing that

*n(n* +1 )(τι + 2) - (n - 1 *)n(n* +1) = n(τι + l)(n+2 - ^∑T), = 3h(ti +1), we have vn+l = θn(7t +1 )(τt + 2) ;

and hence 1+3 + 6. . + 1/2n(n + l)= 1/6n(n + l)(n + 2),

as may be at once verified for any particular value of *n.*

Similarly, when the general term is a factorial of the order r, we have

r+1 n(n + l).. (n + r-1) n(n + l).. (n + r)

+ ^^I 1.2^ 7. 1.2 .. (r + 1)

5. If the general term *un* be any rational and integral

function of *n,* we have

, ?i . ?i(?i —1) 0 *nln-*1).. (?i-» + l) *.p*

*un* = «ο + j∆κo + -1--Jδ⅛0... + 3 *L e.*

where the series is continued only up to the term depend­ing on *p,* the degree of the function *un,* for all the subse­quent terms vanish. The series is thus decomposed into a set of series which have each a factorial for the general term, and which can be summed by the last formula ; thus we obtain

. ., (n + l)?i.

«0 + *u1... + un* = (?i +1 *)u0* + i-j—+- *Δ1t0..*

(n + l)>i(w-l). . (tt-j,+I)Λ

+ . 1 . 2. 3. . Qo+1) 0,

which is a function of the degree *p +* 1.

Thus for the before-mentioned series l + 2 + 4 + 8+ .., if it be assumed that the general term *un* is a cubic function of n, and writing down the given terms and forming the differences, 1, 2, 4, 8 ; 1, 2, 4 ; 1,2; 1, we have

Mn = 1 + *y +* ~~+\*2~~ { = j(ra8 + 5n + 6)>as above ) ;

and the sum *u0 + u1. . + un*

(n + l> 1 (?! + l)?i(?i-l) 1 (n + l)n(n-l)(n-2)

—w +1 + j 2 + 123 "+^ 1234 ,

= ?-(?i\* + 2⅛3 +11?!® + 34?! + 24).

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As particular cases we have expressions for the sums of the powers of the natural numbers—

l2 + 22 ...+7i2 = ^n(n+ l)(2τι + l)j l3 + 23.. + *n3 =* |?is(?t +1 )2 (observe that this = (1 + 2 ... + *n*)2) ; and so on.

6. We may, from the expression for the sum of the geometric series, obtain by differentiation other results :

i ι 1 *—T^, ·*

thus 1+7- + ?·3. . .+7∙n-1 = ς gives

1 -r

1 , n , o O , ∕ 1∖ 71-2 *l-rn* 1-7irn-1 + (τt-l)r\*

l+2r + 3∕". . + (τι-l)r =τ-; , = *7z—⅛—'*

*v z dr* 1 -r (1 -r)2

and we might in this way find the sum *u0 + nlr.. . + uttrn,* where *un* is any rational and integral function of *n.*

7. The expression for the sum *u0 + ul. .*. +«„ of an in­definite number of terms will in many cases lead to the sum of the infinite series *u0 + u-l..*. ; but the theory of infinite series requires to be considered separately. Often in dealing apparently with an infinite series *u0 + u1 + ...* we consider rather an indefinite than an infinite series, and are not in any wise really concerned with the sum of the series or the question of its convergency : thus the equation

(1+w,+5⅛Z‰ + . .)(1+^+⅛L<LV+ ..) =i+<m+.>+ (~~-+")(∞÷"-υ~~~~j~~~~,~~~~+~~~~..~~

really means the series of identities (??i + 7t)=*m + n*

(?!! + 7l)(7Zl + 71 - 1 ) 77l(ni-l) *??! H* 7l(>l-l)

Ï—2 1—2— + 1 Ï + ^^Ï—*2~’&C'’*

obtained by multiplying together the two series of the left-hand side. Again, in the method of generating func­tions we are concerned with an equation *ψ(f) =* √lθ + *Λ1t... + Antn+* .., where the function *φ(t)* is used only to ex­press the law of formation of the successive coefficients.

It is an obvious remark that, although according to the original definition of a series the terms are considered as arranged in a determinate order, yet in a finite series