them positive we attend only to the positive quadrant of the plane, but otherwise to the whole plane ; and we have thus a doubly infinite system or lattice-work of points. We may imagine a boundary depending on a parameter *T* which for *T=∞* is at every point thereof at an infinite distance from the origin ; for instance, the boundary may be the circle *x2 + y2* = 7τ, or the four sides of a rectangle, ,r=±α7τ, *y=±βT.* Suppose the form is given and the value of *T,* and let the sum ∑‰ι n be understood to denote the sum of those terms *um,ιn* which correspond to points within the boundary, then, if as *T* increases without limit the sum in question continually approaches a determinate limit (dependent, it may be, on the form of the boundary), *for such form of boundary* the series is said to be conver­gent, and the sum of the doubly infinite series is the afore­said limit of the sum ∑‰j7l. The condition of convergency may be otherwise stated : it must be possible to take *T* so large that the sum ∑‰1 w for all terms ‰j n which correspond to points outside the boundary shall be as small as we please.

It is easy to see that, if the terms n be all of them positive, and the series be convergent for any particular form of boundary, it will be convergent for any other form of boundary, and the sum will be the same in each case. Thus, let the boundary be in the first instance the circle æ2 + *y2 = T* ; by taking *T* sufficiently large the sum ∑‰ι n for points outside the circle may be made as small as we please. Consider any other form of boundary—for in­stance, an ellipse of given excentricity,—and let such an ellipse be drawn including within it the circle *x2 + y2 = T.* Then the sum ∑κm,n for terms ‰,w corresponding to points outside the ellipse will be smaller than the sum for points outside the circle, and the difference of the two sums —that is, the sum for points outside the circle and inside the ellipse—will also be less than that for points outside the circle, and can thus be made as small as we please. Hence finally the sum ∑⅝ι,,, whether restricted to terms «Μ, n corresponding to points inside the circle or to terms corresponding to points inside the ellipse, will have the same value, or the sum of the series is independent of the form of the boundary. Such a series, viz., a doubly infinite convergent series of positive terms, is said to be absolutely convergent ; and similarly a doubly infinite series of positive and negative terms which is convergent when the terms are all taken as positive is absolutely convergent.

20. We have in the preceding theory the foundation of

the theorem (§ 17) as to the product of two absolutely convergent series. The product is in the first instance expressed as a doubly infinite series ; and, if we sum this for the boundary *x + y — T,* this is in effect a summation of the series -wθvθ + (w0v1 + w1wθ) + . . , which is the product of the two series. It may be further remarked that, starting with the doubly infinite series and summing for the rectangular boundary *x = aT, y = βT,* we obtain the sum as the product of the sums of the two single series. For series not absolutely convergent the theorem is not true. A striking instance is given by Cauchy : the series 1 - + ζ∕3 - χ11 + · · \*s ∞nvθrgθnt and has a calcul­

able sum, but it can be shown without difficulty that

its square, viz., the series 1 /3 + (~7⅞ + 2) - · , ’

is divergent. v ∖ -

21. The case where the terms of a series are imaginary comes under that where they are real. Suppose the general term is *pn + qni,* then the series will have a sum, or will be convergent, if and only if the series having for its general term *pn* and the series having. for its general term *qn* be each convergent ; then the sum = sum of first series + « into sum of second series. The notion of absolute conver-

**β ∕**

gence will of course apply to each of the series separately ; further, if the series having for its general term the modulus *\*Jp2n + Q2∏* be convergent (that is, absolutely convergent, since the terms are all positive), each of the component series will be absolutely convergent ; but the condition is not necessary for the convergence, or the absolute convergence, of the two component series respectively.

22. In the series thus far considered the terms are actual numbers, or are at least regarded as constant ; but we may have a series *uQ + u1 + u2* +.. where the successive terms are functions of a parameter *z* ; in particular we may have a series α0 + *a1z + a2z2 .* . arranged in powers of *z.* It is in view of a complete theory *necessary* to consider *z* as having the imaginary value *x + iy* = r(cos *φ + i* sin φ). The two component series will then have the general terms *anrn* cos *nφ* and *anrn* sin *nφ* respectively ; accordingly each of these series will be absolutely convergent for any value whatever of *φ,* provided the series with the general term *anrn* be absolutely convergent. Moreover, the series, if thus absolutely convergent for any particular value *R* of r, will be absolutely convergent for any smaller value of r, that is, for any value of *x + iy* having a modulus not exceeding *R ;* or, representing as usual *x + iy* by the point whose rect­angular coordinates are æ, *y,* the series will be absolutely con­vergent for any point whatever inside or on the circumfer­ence of the circle having the origin for centre and its radius *= R.* The origin is of course an arbitrary point. Or, what is the same thing, instead of a series in powers of *z,* we may consider a series in powers of *z - c* (where *c* is a given imaginary value = α + *βί).* Starting from the series, we may within the aforesaid limit of absolute convergency con­sider the series as the definition of a function of the vari­able *z* ; in particular the series may be absolutely conver­gent for every finite value of the modulus, and we have then a function defined for every finite value whatever *x + iy* of the variable. Conversely, starting from a given function of the variable, we may inquire under what conditions it admits of expansion in a series of powers of 2 (or *z —* c), and seek to determine the expansion of the function in a series of this form. But in all this, however, we are tra­velling out of the theory of series into the general theory of functions.

. 23. Considering the modulusr as a given quantity and the several powers of *r* as included in the coefficients, the com­ponent series are of the forms *a0* + α1cos *φ* + σ2cos *2φ +. .* and α1sin<∕> + α2sin2<∕> + . . respectively. The theory of these trigonometrical or multiple sine and cosine series, and of the development, under proper conditions, of an arbitrary function in series of these forms, constitutes an important and interesting branch of analysis.

24. In the case of a real variable *z,* we may have a series *ao ÷ aιz +* · · ? where the series *a0 + a1 + a2.*. is a diver­

gent series of decreasing positive terms (or as a limiting case where this series is 1 + 1 + 1 . .). For a value of *z* inferior but indefinitely near to ± 1, say 2=±(1 - ∈), where e is indefinitely small and positive, the series will be convergent and have a determinate sum *φ(z),* and we may write ≠( ± 1 ) to denote the limit of *φ(* ± (1 - ∈) ) as e diminishes to zero ; but unless the series be convergent for the value *z=* ± 1 it cannot for this value have a sum, nor consequently a sum = ≠( ± 1). For instance, let the series be *z +* + j . .,

which for values of *z* between the limits ± 1 (both limits excluded) = - log(l -2). For 2 = + 1 the series is divergent and has no sum ; but for 2 = 1 — c as c dimi­nishes to zero we have - log e and (1 - e) + ^(1 - e)2..., each positive and increasing without limit; for 2= - 1 the series l-i + ^- ^..is convergent, and we have *at*