tion of the problem of rectification (see fig. 2). ACB being a semicircle whose centre is O, and AC the arc to be rectified, he produced AB to D, making BD equal to the radius, joined DC, and produced it to meet the tangent at A in E ; and then his assertion (not established by him) was that AE was nearly equal to the arc AC, the error being in defect. For the purposes of the calculator a solution erring in excess was also required, and this Snell gave by slightly varying the former construction. Instead of producing AB

(see fig. 3) so that BD

was equal to *r,* he pro­

duced it only so far that,

when the extremity D' was

joined with C, the part of

D'C outside the circle was equal to *r* ; in other words, by a non-Euclidean construction he trisected the angle AOC, for it is readily seen that, since FD' = FO = OC, the angle FOB = 1/3AOC.@@1 This couplet of constructions is as im­portant from the calculator’s point of view as it is interest­ing geometrically. To compare it on this

score with the fundamental proposition of

Archimedes, the latter must be put into

a form similar to Snell’s. AMC being an

arc of a circle (see fig. 4) whose centre is

O, AC its chord, and HK the tangent

drawn at the middle point of the arc and

bounded by OA, OC produced, then, according to Archi­medes, AMC<HK but >AC. In modern trigonometrical notation the propositions to be compared stand as follows :—

2tan1/2*θ>θ>*2sin1/2*θ* (Archimedes); tan 1/3*θ* + 2sin1/3*θ*>*θ*> 3sin*θ*/2 + cos*θ* (Snell).

It is readily shown that the latter gives the best approxi­mation to *θ* ; but, while the former requires for its applica­tion a knowledge of the trigonometrical ratios of only one angle (in other words, the ratios of the sides of only one right-angled triangle), the latter requires the same for two angles, *θ* and 1/3*θ.* Grienberger, using Snell’s method, cal­culated the ratio correct to 39 fractional places.@@2 Huy­gens, in his *De Circuli Magnitudine Inventa,* 1654, proved the propositions of Snell, giving at the same time a number of other interesting theorems, for example, two inequalities which may be written as follows @@3—

chd *θ +* ⅛(chd *θ* - sin *θ') > θ* > chd *θ* + i(chd *θ -* sin *θ).*

2 chd 0 + 3 sin ø 3v ' 3ξ 7

As might be expected, a fresh view of the matter was taken by Descartes. The problem he set himself was the exact converse of that of Archimedes. A given straight line being viewed as equal in

length to the circumference of a

circle, he sought to find the dia­

meter of the circle. His con­

struction is as follows (see fig. 5).

Take AB equal to one-fourth of

the given line ; on AB describe a

square ABCD ; join AC ; in AC

produced find, by a known process,

a point C1 such that, when C1B1

is drawn perpendicular to AB pro­

duced and C1D1 perpendicular to BC produced, the rect­angle BC1 will be equal to 1/4ABCD ; by the same process find a point C2 such that the rectangle B1C2 will be equal to 1/4BC1 ; and so on *ad infinitum.* The diameter sought is the

straight line from A to the limiting position of the series of B’s, say the straight line AB∞. As in the case of the process of Archimedes, we may direct our attention either to the infinite series of geometrical operations or to the corresponding infinite series of arithmetical opera­tions. Denoting the number of units in AB by 1/4*c,* we can express BB1, B1B2, . . . in terms of 1/4*c,* and the identity AB∞ = AB + BB1 + B1B2 + ... gives us at once an expres­sion for the diameter in terms of the circumference by means of an infinite series.@@4 The proof of the correctness of the construction is seen to be involved in the following theorem, which serves likewise to throw new light on the subject :—AB being any straight line whatever, and the above construction being made, then AB is the diameter of the circle circumscribed by the square ABCD (self-evi­dent), AB1 is the diameter of the circle circumscribed by the regular 8-gon having the same perimeter as the square, AB2 is the diameter of the circle circumscribed by the regular 16-gon having the same perimeter as the square, and so on. Essentially, therefore, Descartes’s process is that known later as the process of *isoperimeters,* and often attributed wholly to Schwab.@@5

In 1655 appeared the *Arithmetica Infinitorum* of Wallis, where numerous problems of quadrature are dealt with, the curves being now represented in Cartesian coordinates, and algebra playing an important part. In a very curious manner, by viewing the circle *y* = (1 *-x*2)1/2 as a member of the series of curves *y =* (l*-x*2)1*, y =* (1*-x*2)2*,* &c., he was led to the proposition that four times the reciprocal of the ratio of the circumference to the diameter is equal to

3.3.5.5. 7.7.9...

2.4.4. 6.6.8. 8.. .’

and; the result having been communicated to Lord Broun- ker, the latter discovered the equally curious equivalent expression ι ÷12 9->

2 + — 52

+2+k...

The work of Wallis had evidently an important influence on the next notable personality in the history of the sub­ject, James Gregory, who lived during the period when the higher algebraic analysis was coming into power, and whose genius helped materially to develop it. He had, however, in a certain sense one eye fixed on the past and the other towards the future. His first contribution@@6 was a variation of the method of Archimedes. The latter, as we know, calculated the perimeters of successive polygons, passing from one polygon to another of double the number of sides ; in a similar manner Gregory calculated the areas. The general theorems which enabled him to do this, after a start had been made, are

*A*2*η=∖∕AηA'η* (Snell’s *Cyclom.),*

*., 2AnA'n 2A'nAin .r, .*

-4≈∙=⅛⅞.or⅛⅜∕0rβs01yλ where *An, A'n* are the areas of the inscribed and the circum­scribed regular *n*-gons respectively. He also gave approxi­mate rectifications of circular arcs after the manner of Huy­gens; and, what is very notable, he made an ingenious and, according to Montucla, successful attempt to show that quadrature of the circle by a Euclidean construction was impossible.@@7 Besides all this, however, and far beyond it in importance, was his use of infinite series. This merit he shares with his contemporaries Mercator, Newton, and Leibnitz, and the exact dates of discovery are a little un­certain. As far as the circle-squaring functions are con-

@@@1 It is thus manifest that by his first construction Snell gave an approximate solution of two great problems of antiquity.

@@@2 *Elementa Trigonometrica,* Rome, 1630 ; Glaisher, *Messenger of Math.,* iii. p. 35 *sq.*

@@@3 See Kiessling’s edition of the *De Circ. Magn. Inν.,* Flensburg, 1869 ; or Pirie’s tract on *Geometrical Methods of Approx. to the Value of π,* London, 1877.

@@@4 See Euler, “ Annotationes in Locum quendam Cartesii, in *Nov. Comm. Acad. Petrop.,* viii.

@@@5 Gergonne, *Annales de Math.,* vi.

@@@6 See *Vera Circuli et Hyperbolæ Quadratura,* Padua, 1667 ; and the *Appendicula* to the same in his *Exercitationes Geometricae,* London, 1668.

@@@7 *Penny Cyclop.,* xix. p. 187.