cerned, it would seem that Gregory was the first (in 1670) to make known the series for the arc in terms of the tangent, the series for the tangent in terms of the arc, and the secant in terms of the arc; and in 1669 Newton showed to Barrow a little treatise in manuscript containing the series for the arc in terms of the sine, for the sine in terms of the arc, and for the cosine in terms of the arc. These discoveries formed an epoch in the history of mathe­matics generally, and had, of course, a marked influence on after investigations regarding circle-quadrature. Even among the mere computers the series

*θ=*tan *θ -* 1/3 tan3 *θ* + 1/5 tan5 *θ-...,* specially known as Gregory’s series, has ever since been a necessity of their calling.

The calculator’s work having now become easier and more mechanical, calculation went on apace. In 1699 Abraham Sharp, on the suggestion of Halley, took Gregory’s series, and, putting tan *θ* = 1/3 √3, found the ratio equal to

√12(1 - 1/3.3 + \*\*\* - \*\*\* ),

from which he calculated it correct to 71 fractional places.@@1 About the same time Machin calculated it correct to 100 places, and, what was of more importance, gave for the ratio the rapidly converging expression,

16/ 1 , 1 1 ∖ 4 ∕ l l ∖

5∖1 3.52 + 5.54 7.5β+',√ 239∖i 3.2392 + 5.2394 ” which long remained without explanation.@@2 Fautet de Lagny, still using tan 30°, advanced to the 127th place.@@3

Euler took up the subject several times during his life, effecting mainly improvements in the theory of the various series.@@4 With him, apparently, began the usage of denot­ing by π the ratio of the circumference to the diameter.@@5 The most important publication, however, on the subject in the 18th century was a paper by Lambert,@@6 read before the Berlin Academy in 1761, in which he demonstrated the irrationality of *π.* The general test of irrationality which he established is that, if

**«1**

**T^-Uα2**

2 ⅛±...

be an interminate continued fraction, *a*1, *a*2..., *b*1, *b2 ...*

be integers, , ^,... be proper fractions, and the value of

h 4 . . . α,

every one of the interminate continued fractions

*fτ* β β 1 — \*5

,— ,... be < 1, then the given continued fraction

δ2±..., . . .

represents an irrational quantity. If this be applied to the right-hand side of the identity

mm , tan — = — ∞ ,

*η n--^- rni* 3n- =—

5n -...,

it follows that the tangent of every arc commensurable with the radius is irrational, so that, as a particular case, an arc of 45°, having its tangent rational, must be incom­

mensurable with the radius; that is to say, π/4 is an incom­mensurable number.@@7 This incontestable result had no effect, apparently, in repressing the π-computers. Vega

in 1789, using series like Machin’s, viz., Gregory’s series and the identities

π/4 = 5 tan-1 1/7 + 2 tan-1 3/79 (Euler, 1779),

π/4 = tan-1 1/7 + 2 tan-1 1/3 (Hutton, 1776),

neither of which was nearly so advantageous as several found by Hutton, calculated *π* correct to 136 places.@@8 This achievement was anticipated or outdone by an un­known calculator, whose manuscript was seen in the Rad­cliffe Library, Oxford, by Baron von Zach towards the end of the century, and contained the ratio correct to 152 places. More astonishing still have been the deeds of the π-computers of the 19th century. A condensed record compiled by Mr Glaisher *(Messenger of Math.,* ii. p. 122) is as follows :—

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Date. | Computer. | No. of No. of fr.digits fr.digits calcd. correct. | | Place of Publication. |
| 1842 | Rutherford | 208 | 152 | *Trans. Roy. Soc.,* Lond., 1841, p. 283. |
| 1844 | Dase | 205 | 200 | *Crelle's Journ.,* xxvii. p. 198. |
| 1847 | Clausen ... | 250 | 248 | *Astron. Nachr.,* xxv. col. 207. |
| 1853 | Shanks ... | 318 | 318 | *Proc.Roy. Soc.,* Lond.,1853, p. 273. |
| 1853 | Rutherford | 440 | 440 | *Ibid*. |
| 1853 | Shanks ... | 530 |  | *Ibid.* |
| 1853 | Shanks ... | 607 |  | W. Shanks, *Rectification of the Circle,* London, 1853. |
| 1853 | Richter ... | 333 | 330 | *Grunert’s Archiv,* xxi. p. 119. |
| 1854 | Richter ... | 400 | 330 | *Ibid.,* xxii. p. 473. |
| 1854 | Richter ... | 400 | 400 | *Ibid.,* xxiii. p. 476. |
| 1854 | Richter ... | 500 | 500 | *Ibid.,* XXV. p. 472. |
| 1873 | Shanks ... | 707 |  | *Proc. Roy. Soc.,* Lond., xxi. |

By these computers Machin’s identity, or identities ana­logous to it, *e.g.,*

π/4 = tan-1 1/2 + tan-1 1/5 + tan-1 1/8,

π/4 = 4 tan-1 1/5 - tan-1 7 1/70 + tan-1 1/99,

and Gregory’s series were employed.@@9

A much less wise class than the π-computers of the 19th century are the pseudo-circle-squarers, or circle-squarers technically so called, that is to say, persons who, having obtained by illegitimate means a Euclidean construction for the quadrature or a finitely expressible value for *π,* insist on using faulty reasoning and defective mathematics to establish their assertions. Such persons have flourished at all times in the history of mathematics ; but the interest at­taching to them is more psychological than mathematical.@@10

It is of recent years that the most important advances in the theory of circle-quadrature have been made. In 1873 Hermite proved that the base ε of the Napierean logarithms cannot be a root of a rational algebraical equation of any degree.@@11 To prove the same proposition regarding π is to prove that a Euclidean construction for circle-quadrature is impossible. For in such a construction every point of the figure is obtained by the intersection of two straight lines, a straight line and a circle, or two circles ; and, as this implies that, when a unit of length is introduced, numbers employed, and the problem trans­formed into one. of algebraic geometry, the equations to be solved can only be of the first or second degree, it follows that the equation to which we must be finally led is a rational equation of even degree. Hermite@@12 did not

@@@1 See Sherwin’s Math. Tables, London, 1705, p. 59.

@@@2 See W. Jones, Synopsis Palmariorum Matheseos, London, 1706;

Maseres, Scriptores Logarithmici, London, 1791-96, vol. iii. pp. 159 sq. ; Hutton, Tracts, vol. i. p. 266.

@@@3 See Hist. de l’Acad., Paris, 1719 ; 7 appears instead of 8 in the 113th place.

@@@4 Comment. Acad. Petrop., ix., xi. ; Nov. Comm. Ac. Pet., xvi. ; Nova Acta Acad. Pet., xi.

@@@5 Introd. in Analysin Infin., Lausanne, 1748, chap, viii.

@@@6 Mém. sur quelques propriétés remarquables des quantités transcend­

antes, circulaires, et logarithmiques.

@@@7 See Legendre, Éléments de Geometrie, Paris, 1794, note iv. ; Schlö­

milch, Handbuch d. algeb. Analysis, Jena, 1851, chap. xiii.

@@@8 Noca Acta Petrοp., ix. p. 41; Thesaurus Logarithm. Completus, p. 633.

@@@9 On the calculations made before Shanks, see Lehmann, “Beitrag zur Berechnung der Zahl π,” in Grunert's Archiv, xxi. pp. 121-174.

@@@10 See Montucla, Hist. des rech. sur la quad. du cercle, Paris, 1754, 2d ed. 1831 ; De Morgan, Budget of Paradoxes, London, 1872.

@@@11 “Sur la fonction exponentielle,’’ Comptes Rendus, Paris, lxxvii. pp. 18, 74, 226, 285.

@@@12 See Crelle s Journal, lxxvi. p. 342.