SURFACE, CONGRUENCE, COMPLEX. In the article Curve the subject was treated from an historical point of view for the purpose of showing how the leading ideas of the theory were successively arrived at. These leading ideas apply to surfaces, but the ideas peculiar to surfaces are scarcely of the like fundamental nature, being rather developments of the former set in their application to a more advanced portion of geometry; there is conse­quently less occasion for the historical mode of treatment. Curves in space were briefly considered in the same article, and they will not be discussed here ; but it is proper to refer to them in connexion with the other notions of solid geometry. In plane geometry the elementary figures are the point and the line; and we then have the curve, which may be regarded as a singly infinite system of points, and also as a singly infinite system of lines. In solid geo­metry the elementary figures are the point, the line, and the plane; we have, moreover, first, that which under one aspect is the curve and under another aspect the develop­able (or torse), and which may be regarded as a singly in­finite system of points, of lines, or of planes; and secondly, the surface, which may be regarded as a doubly infinite system of points or of planes, and also as a special triply infinite system of lines. (The tangent lines of a surface are a special complex.) As distinct particular cases of the first figure we have the plane curve and the cone, and as a particular case of the second figure the ruled surface, regulus, or singly infinite system of lines ; we have, be­sides, the congruence or doubly infinite system of lines and the complex or triply infinite system of lines. And thus crowds of theories arise which have hardly any ana­logues in plane geometry; the relation of a curve to the various surfaces which can be drawn through it, and that of a surface to the various curves which can be drawn upon it, are different in kind from those which in plane geometry most nearly correspond to them,—the relation of a system of points to the different curves through them and that of a curve to the systems of points upon it. In particular, there is nothing in plane geometry to correspond to the theory of the curves of curvature of a surface. Again, to the single theorem of plane geometry, that a line is the shortest distance between two points, there correspond in solid geometry two extensive and difficult theories,—that of the geodesic lines on a surface and that of the minimal surface, or surface of minimum area, for a given boundary. And it would be easy to say more in illustration of the great extent and complexity of the subject.

*Surfaces in General; Torses, &c.*

1. A surface may be regarded as the locus of a doubly in­finite system of points,—that is, the locus of the system of points determined by a single equation *U*=(\*)(*x, y, z,* 1)*n*, = 0, between the Cartesian coordinates (to fix the ideas, say rectangular coordinates) *x, y, z*; or, if we please, by a single homogeneous relation *U*=(\*)(*x, y, z, w*)*n*, = 0, between the quadriplanar coordinates *x, y, z, w.* The degree *n* of the equation is the order of the surface; and this defini­tion of the order agrees with the geometrical one, that the order of the surface is equal to the number of the inter­sections of the surface by an arbitrary line. Starting from the foregoing point definition of the surface, we might develop the notions of the tangent line and the tangent plane; but it will be more convenient to consider the sur­face *ab initio* from the more general point of view in its relation to the point, the line, and the plane.

2. Mention has been made of the plane curve and the cone; it is proper to recall that the *order* of a plane curve is equal to the number of its intersections by an arbitrary line (in the plane of the curve), and that its *class* is equal to the number of tangents to the curve which pass through

an arbitrary point (in the plane of the curve). The cone is a figure correlative to the plane curve : corresponding to the plane of the curve we have the vertex of the cone, to its tangents the generating lines of the cone, and to its points the tangent planes of the cone. But from a differ­ent point of view we may consider the generating lines of the cone as corresponding to the points of the curve and its tangent planes as corresponding to the tangents of the curve. From this point of view we define the order of the cone as equal to the number of its intersections (generating lines) by an arbitrary plane through the vertex, and its class as equal to the number of the tangent planes which pass through an arbitrary line through the vertex. And in the same way that a plane curve has singularities (singular points and singular tangents) so a cone has singularities (singular generating lines and singular tangent planes).

3. Consider now a surface in connexion with an arbi­trary line. The line meets the surface in a certain number of points, and, as already mentioned, the *order* of the sur­face is equal to the number of these intersections. We have through the line a certain number of tangent planes of the surface, and the *class* of the surface is equal to the number of these tangent planes.

But, further, through the line imagine a plane; this meets the surface in a curve the order of which is equal (as is at once seen) to the order of the surface. Again, on the line imagine a point; this is the vertex of a cone cir­cumscribing the surface, and the class of this cone is equal (as is at once seen) to the class of the surface. The tangent lines of the surface which lie in the plane are nothing else than the tangents of the plane section, and thus form a singly infinite series of lines; similarly, the tangent lines of the surface which pass through the point are nothing else than the generating lines of the circumscribed cone, and thus form a singly infinite series of lines. But, if we consider those tangent lines of the surface which are at once in the plane and through the point, we see that they are finite in number; and we define the *rank* of a surface as equal to the number of tangent lines which lie in a given plane and pass through a given point in that plane. It at once follows that the class of the plane section and the order of the circumscribed cone are each equal to the rank of the surface, and are thus equal to each other. It may be noticed that for a general surface *U*=(\*)(*x, y, z, w*)*n*, = 0, of order *n* without point singularities the rank is *a*, = *n(n* - 1), and the class is *n', = n(n* - l)2; this implies (what is in fact the case) that the circumscribed cone has line singularities, for otherwise its class, that is the class of the surface, would be *a(a* - 1 ), which is not = *n(n* - 1 )2.

4. In the last preceding number the notions of the tangent line and the tangent plane have been assumed as known, but they require to be further explained in refer­ence to the original point definition of the surface. Speak­ing generally, we may say that the points of the surface consecutive to a given point on it lie in a plane which is the tangent plane at the given point, and conversely the given point is the point of contact of this tangent plane, and that any line through the point of contact and in the tangent plane is a tangent line touching the surface at the point of contact. Hence we see at once that the tangent line is any line meeting the surface in two consecutive points, or—what is the same thing—a line meeting the surface in the point of contact counting as two intersec­tions and in *n* - 2 other points. But, from the foregoing notion of the tangent plane as a plane containing the point of contact and the consecutive points of the surface, the passage to the true definition of the tangent plane is not equally obvious. A plane in general meets the surface of the order *n* in a curve of that order without double points ; but the plane may be such that the curve has a