already indicated for “ harmonizing the angles of trigonometrical figures,” of which it is merely an expansion from single ligures to great groups.

The rigorous treatment would, however, have involved the simultaneous solution of about 4000 equations between 9230 un­known quantities, which was quite impracticable. The triangula­tion was therefore divided into sections for separate reduction, of which the most important were the live between the meridians of 67° and 92° (see fig. 1, p. 696), consisting of four quadrilateral figures and a trigon, each comprising several chains of triangles and some base-lines. This arrangement had the advantage of enabling the final reductions to be taken in hand as soon as convenient after the completion of any section, instead of being postponed until all were completed. It was subject, however, to the condition that the sections containing the best chains of triangles were to be first reduced ; for, as all chains bordering contiguous sections would necessarily be “ fixed ” as a part of the section first reduced, it was obviously desirable to run no risk of impairing the best chains by forcing them into adjustment with others of inferior quality. It happened that both the north-east and the south-west quadrilaterals contained several of the older chains ; their reduction was therefore made to follow that of the collateral sections containing the modern chains.

But the reduction of each of these great sections was in itself a very formidable undertaking, necessitating some departure from a purely rigorous treatment. For the chains were largely composed of polygonal networks and not of single triangles only as assumed in the illustration, and therefore cognizance had to be taken of a number of “side” and other geometrical equations of condition, which entered irregularly aud caused great entanglement. Equa­tions 17 and 18 of the illustration are of a simple form because they have a single geometrical condition to maintain, the triangular, which is not only expressed by the simple and symmetrical equation *x + y +z=*0*,* but—what is of much greater importance—recurs in a regular order of sequence that materially facilitates the general solution. Thus, though the calculations must in all cases be very numerous and laborious, rules can be formulated under whieh they can be well controlled at every stage and eventually brought to a successful issue. The other geometrical conditions of networks are expressed by equations which are not merely of a more complex form but have no regular order of sequence, for the networks pre­sent a variety of forms ; thus their introduction would cause much entanglement and complication, and greatly increase the labour of the calculations and the chances of failure. Wherever, therefore, any compound figure occurred, only so much of it as was required to form a chain of single triangles was employed. The figure having previously been made consistent, it was immaterial what part was employed, but the selection was usually made so as to introduce the fewest triangles. The triangulation for final simultaneous reduction was thus made to consist of chains of single triangles only; but all the included angles were “fixed” simultaneously. The excluded angles of compound figures were subsequently har­monized with the fixed angles, which was readily done for each figure *per se.*

This departure from rigorous accuracy was not of material im­portance, for the angles of the compound figures excluded from the simultaneous reduction had already, in the course of the several independent figurai adjustments, been made to exert their full in­fluence on the included angles. The figurai adjustments had, how­ever, introduced new relations between the angles of different figures, causing their weights to increase *cæteris paribus* with the number of geometrical conditions satisfied in each instance. Thus, suppose *w* to be the average weight of the *t* observed angles of any figure, and *n* the number of geometrical conditions presented for satisfaction ; then the average weight of the angles after adjustment may be taken as *w*.*t/t-n,* the factor thus being 1·5 for a triangle, 1·8 for

a hexagon, 2 for a quadrilateral, 2·5 for the network around the Sironj base-line, &c.

In framing the normal equations between the indeterminate factors λ for the final simultaneous reduction, it would have greatly added to the labour of the subsequent calculations if a separate weight had been given to each angle, as was done in the primary figurai reductions ; this was obviously unnecessary, for theoretical requirements would now be amply satisfied by giving equal weights to all the angles of each independent figure. The mean weight that was finally adopted for the angles of each group was therefore taken as *t*

*w.ρ.t/t-n*

*p* being the modulus already indicated in section 12.

The second of the two processes for applying the method of minimum squares having been adopted, the values of the errors *y* and *z* of the angles appertaining to any, the *p*th, triangle were finally expressed by the following equations, which are derived from (18) by substituting *u for* the reciprocal final mean weight as above determined :—

*Vp*=⅛ [(2⅛> - ¾ - ⅛)χ] Ί

„ 1 (19)∙

⅞=⅞[(2⅛-⅝-⅛,)λ]J

The most laborious part of the calculations was the construction and solution of the normal equations between the factors λ. On this subject a few hints are desirable, because the labour involved is liable to be materially influenced by the order of sequence adopted in the construction. The normal equations invariably take the form of (4), the coefficients on the diagonal containing summations of squares of the coefficients in the primary equations, while those above and below contain summations of products of the primary quantities, such that the coefficient of the *p*th λ in the *q*th equation is the same as that of the *q*th λ in the *p*th equation. In practice, as any single angular error only enters a few of the primary equa­tions of condition, many of the coefficients vanish, both in the primary and in the normal equations ; and it is an object of great importance so to arrange the normal equations that most blanks shall occur *above* and fewest blanks *between* the significant values on each vertical line of coefficients ; in other words, the significant values above and below the diagonal should lie as closely as possible to the diagonal, every value on which is always significant. This advantage is secured when the primary equations are arranged in groups in which each contains a number of angular errors in common and as many as possible of those entering the group on each side. Thus the arrangement must follow the natural succes­sion of the chains of triangles rather than the characteristics of the primary equations ; if, for example, all the side equations were grouped together, and all the latitude equations, and so on, great entanglement would arise in the solution of the normal equations, enormously increasing the labour and the chances of failure. The best arrangement was found to be to group the side and the three geodetic equations of each circuit together in the order of sequence of the meridional chains of triangles, and then to introduce the side equations connecting base-lines between the groups with which they had most in common.

The following table (II.) gives the number of equations of condi­tion and unknown quantities—the angular errors—in the five great sections of the triangulation, which were respectively included in the simultaneous general reductions and relegated to the subse­quent adjustments of each figure *per se :—*

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Section. | Simultaneous. | | | External Figurai. | | | | | |
| Equations. | | Angular  Errors. | Equations. | | | | Angular  Errors. | No. of Figures. |
| Circuit and Base­line. | Tri­  angular. | Tri­  angular. | Central. | Side. | Toto- partial. |
| 1. N.W. Quad. .. | 23 | 550 | 1650 | 267 | 104 | 152 | 6 | 761 | 110 |
| 2. S.E. Quad. .. | 15 | 277 | 831 | 164 | 64 | 92 | 2 | 476 | 68 |
| 3. N.E. Quad. .. | 4 | 573 | 1719 | 112 | 56 | 69 | 0 | 341 | 50 |
| 4. Trigon | 22 | 303 | 909 | 192 | 79 | 101 | 2 | 547 | 77 |
| 5. S.W. Quad. .. | 24 | 172 | 516 | 83 | 32 | 52 | 1 | 237 | 40 |

The magnitudes of the 2481 angular errors determined simultane­ously in the first two sections were very small, 2240 being under 0"·1, 205 between 0"·1 and 0"·1, 33 between 0"·2 and 0"·3, 2 between 0''·3 and 0"·4, and 1 between 0"·4 and 0"·5. In the third section, which contained a number of old chains, executed with instruments inferior to the 2 and 3 foot theodolites, they were larger: 780 were under 0"·1, 911 between 0"·1 and 1"·0, 27 between 1"·0 and 2''·0, and 1 between 2"·0 and 2"·1. Thus the corrections to the angles were generally very minute, rarely exceeding the theoretical probable errors of the angles, and therefore applicable without taking any liberties with the facts of observation.

18. *Theoretical Error of any Function of Angles of a Geometrically corrected Triangulation.—*The investigation of such theoretical errors was no easy matter. When first essayed it was generally assumed by mathematicians in England that any attempt to exhibit the theoretical error by a purely algebraical process soon led to results of in­tolerable complexity, so that it was desirable to introduce numbers as soon as possible for every symbol except the absolute terms of the geometrical or primary equations of condition. But on continuing the algebraical process cer­tain relations were found to exist between the coefficients of the indeterminate factors in the normal equations of the minimum square method and the coefficients of the un­known quantities in the primary equations of condition, which enormously simplified the process and led to a general algebraical expression of no great complexity ; it was also found that, the number of primary equations being », the