labour of calculation by the formula was reduced to an nth of that involved by resorting at once to numbers.

Let *F* be any function whatever of the corrected angles (*X*1 - *x*1), *X*2 *-* *x*2,),. . . of a trigonometrical figure ; let

*, \_ clF , \_ dF ,*

*-'1~dX2 ^'2 dX2' · ■ ’*

also let *u*1, *u*2, .... symbols hitherto employed to represent the rela­tive reciprocal weights of the observed angles *X*1, *X*2, . . . , in future represent absolute measures of precision, the *p.e.2* of the observed angles ; then the following formula expresses the *p.e.* of any func­tion of the corrected angles rigorously :—

,,z,2,,f p *( + [fa∙U]ttfa.u]Aa + tfb.u]Ab+ ... + [fn.u]AA ∖*

*=fSf-l + lP>.u][[fa.u]Ba + [fb.u]Bi + ...+[fn.u]BA* l(20).

V + *[f>ι. u]* {iy⅛. u]*Xra* + [/ό. *u]* ivi + ...+[/». *u]* 2Vn}' *J* The symbols *a, b, . . . n* have the same signification as in (3) to (6) of section 13. *A, B,. . . N* are coefficients which must be de­termined in the process of solving the normal equations as follows :—

*∖a = Aaea + Aι,eb + · · · +*

λ⅛ *= Baea + Bbe∣, + . . . + Bnen* I (21)

λn = *Naea + Nbob+ ∙ · · + NHen J*

where the coefficient represented by any two letters in one order is identical with that represented by the same letters in the reverse order ; thus *An = Na.* Hence to find the *p.e.* of any angle, as (*X*1 - *x*1), in a single triangle we have

*f1* = l, and *Aa=r- ,=— — ;*

*j1 [aa*. w] κ1 + *u.2 + u3*

all the other factors vanish, and

*p.e,i* of ( W1 - a∙1)=*u1 - - + + ι- =p.e.n-* of *X1 -p.e.2* of a⅛.

To find the *p.e.* of the ratio *R* of either side to the base,—if *R =* sin (Ar1 - a¾)÷sin (Ar2 - a⅛),

then *fι = P* cot *X1* sin *l",f2-R* cot *X2* sin l",∕3=0,

and *p.e.2* of *R*

7T> · ■>,„ f .<>ιr ..Tr (‰cotx∖-‰cotA2)2 ) \_

=Λ2sm-l {u1c0t2X1+u2cc>t-X2 j (22).

When the function of the corrected angles is the ratio of the terminal to the initial side of an equilateral triangle or a regular quadrilateral or polygon (either of two sides being taken if the figure has an odd number of exterior sides), then, assuming all the angles to be of equal weight, we have the following values of the *ρ.ejs* and the relative weights of the ratios :—

Figure. p.e. Weight. Figure. p.e. Weight.

Triangle ±·82-√*u* sin 1" 1·49

Quadrilateral 1·00 ,, 1·00

Trigon 1·05 ,, 0·90

Tetragon ... 1·15 ,, 0·75

Pentagon ±1∙21√*u* sin 1" 0·68

Hexagon 1·29 ,, 0·60

Heptagon 1∙41 ,, 0·50

Octagon 1,57 ,, 0·41

In ordinary ground seven single triangles will span about as much as two hexagons and the weights of the terminal sides would be as twenty-one by the former to thirty by the latter. In a flat country two quadrilaterals would not span more than one hexagon, giving terminal side weights as five to six ; but in hills a quadrilateral may span as much as any polygon and give a more exact side of con­tinuation. Thus in the Indian Survey polygons predominate in the plains and quadrilaterals in the hills.

The theoretical errors of the lengths and azimuths of the sides, and of the latitudes and longitudes of the stations, at the termini of the chains of triangles or at the circuit closings, might be calculated with the coefficients *a, b,* and *c* of *x, y,* and *z* in the circuit and base-line equations as the *f*s, and the known *p.e'*s of *X,* *Y*, and *Z* and the other data of the figural reductions. Such calculations are, however, much too laborious to be ordinarily under­taken. Thus the exactitude of a triangulation is very generally estimated merely on the evidence of the magni­tudes of the differences between the trigonometrical and the measured lengths of the base-lines ; for, though the combined influence of angular precision and geometrical configuration is what really governs the precision of the results, it is not readily ascertainable, and is therefore generally ignored. But, when questions as to the intrinsic value of a triangulation arise, the theory of errors should

always be appealed to, and its intimations accepted rather than the evidence of base-line discrepancies, which if very small are certainly accidental, and if seemingly large may be no greater than what we should be prepared to expect. Good work has occasionally been redone unnecessarily, and inferior work upheld, because their merits were erroneously estimated. The following formulæ will be found useful in acquiring a fairly approximate knowledge of the magnitude of the errors which theory would lead us to expect, not only in side, but in latitude, longitude, and azimuth also, at the close of any chain of triangles. They indicate rigorously the *p.e'*s at the terminal end of a chain of equilateral triangles of which all the angles have been measured and corrected and are of equal weight ; the results may be made to serve for less symmetrical chains, including networks of varying weight, by the application of certain factors which can be estimated with fair pre­cision in each instance.

Let *c* be the side length, ε the *p.e.* of the angles, *n* the number of triangles, and *R* the ratio (here=l) of the terminal to the initial side, then

*p.e.* of *R =* e sin l"β Vf *n )*

*υ.e.* of azimuth*-c∖Pfιι* I ∕qq∖ ι

sin i"  *·. Ç*

*p.e.* of either coordinate = *ec* —-—V2ît3 + 3n2 + 10?i .....(23).@@1

When the form of the triangles deviates much from the equi­angular, the *p.e.* must be multiplied by a factor increasing up to 1·4 as the angles diminish from 60° to 30°, and a mean value of *c* must be adopted. When the chain is double throughout, the *p.e.* must be diminished by a factor taking cognizance of the greater weight of compound figures than of single triangles. When the chain is composed of groups of angles measured with different in­struments, a separate value of ε must be employed for each group, and the final result obtained from√[*p*.*e*.2]. The *p.e.* of *R* may be determined rigorously for any chain of single triangles, with angles of varying magnitudes and weights, by (22), with little labour of calculation.

19. *Relations between Theoretical Errors of Base-lines and those of a Triangulation.—*These relations have to be investigated in order to ascertain whether the base-lines may be assumed to be errorless in the general reduction of the triangulation ; being fallible quantities, their errors must be included among the unknown quantities to be in­vestigated simultaneously, if their respective *p.e.'*s differ sensibly, or if the *p.e*'s of their ratios are not materially smaller than those of the corresponding trigonometrical ratios. By (23) the *p.e.* of the ratio of any two sides of an equilateral triangle is ε sin 1" √2÷3 ; but the *p.e.* of the ratio of two base-lines of equal length and weight is *η* √2, where *η* is the *p.e.* of either base-line ; thus weight of trigo­nometrical ratio : weight of base-line ratio : *3η2* : ε2 sin2 1 ", or as 3:1 when e= ± 0"·3 and η *=* ± 1·5 millionth parts, which happens generally in the Indian triangulation. But the chains between base-lines were always composed of a large number of triangles, and the average weight of the base-line ratios was about eleven times greater by the direct linear measurements than by the triangulation, even when all the unascertainable constant or accidental errors —as from displacements of mark-stones—which might be latent in the latter were disregarded. Moreover, the base­lines were practically all of the same precision ; they were therefore treated as errorless, and the triangulation was made accordant with them.

If a base-line *AD* be divided at *B* and *C* into three equal sections connected together by equilateral triangles, and every angle has been measured with a *p.e.* = ε, the *p.e.* of any trigonometrical ratio may be put = κ.ε sin 1", *κ* being a coefficient which has two values for each ratio,— the greater value when the triangulation has been carried along one flank of the line, the smaller when along both

@@@1 For an investigation of these formulæ, see Appendix No. 3, vol. vii. of *Account of Operations of Great Τrigonom. Survey,* 1882.