*ment have disappeared through friction is coperiodic with the forces acting on the system.* Hence, if the sea is solicited by a periodic force expressed as a coefficient multiplied by the cosine of an angle which increases proportionately with the time, there results a partial tide, also expressed by the cosine of an angle which increases at the same rate ; but the phase of the angle and the coefficient of the cosine in the expression for the height may be very different from those occurring in the corresponding term of the equilibrium theory. The coefficients and the constants or epochs of the angles in the expressions for the tide are only derivable from observation. The action of the sun and moon is ex­pressible in a converging series of similar cosines ; whence there arise as many partial tides, which by the principle of superposition may be added together to give the total tide at any port. In order to unite the several constants of the partial tides Laplace considers each tide as being produced by a fictitious satellite moving uniformly on the equator. Sir W. Thomson and others have followed La­place in this conception ; but in the present article we shall not do so. The difference of treatment is in reality only a matter of phraseology, and the proper motion of each one of Laplace’s *astres fictifs* is at once derivable from the *argument* (or angle under the sign of cosine), which we shall here associate with the partial tides.

Subsequently to Laplace the most important workers in this field were Sir John Lubbock (senior), Whewell, and Airy. The work of Lubbock and Whewell (see § 34 below) is chiefly remarkable for the coordination and analysis of enormous masses of data at various ports, and the con­struction of trustworthy tide-tables and of cotidal maps. Airy contributed an important review of the whole tidal theory. He also studied profoundly tho theory of waves in canals, and explained the effects of frictional resistances on the progress of tidal and other waves. Of other authors whose work is of great importance we shall speak below.

Amongst all the grand work which has been bestowed on this difficult subject, Newton, notwithstanding his errors, stands out first, and next to him we must rank Laplace. However original any future contribution to the science of the tides may be, it would seem as though it must perforce be based on the work of these two.

A complete list of works bearing on the *theory* of the tides, from the time of Newton down to 1881, is contained in vol. ii. of the *Bibliographie de l' Astronomie* by Houzeau and Lancaster (Brussels, 1882). This list does not con­tain papers on the tides of particular ports, and we are not aware of the existence of any catalogue of works on practical observation, reduction of observations, prediction, and tidal instruments. References are, however, given below to several works on these points.

II. Tide-generating Forces.

§ 5. Investigation of Tide-generating Potential and Forces.

We have already given a general explanation of the nature of tide-generating forces ; we now proceed to a rigorous investigation.

If a planet is attended by a single satellite, the motion of any body relatively to the planet’s surface is found by the process described as reducing the planet’s centre to rest. The planet’s centre will be at rest if every body in the system has impressed on it a velocity equal and opposite to that of the planet’s centre ; and this is accomplished by impressing on every body an accelera­tion equal and opposite to that of the planet’s centre.

Let M, m be the masses of the planet and the satellite ; r the radius vector of the satellite, measured from the planet’s centre ; p the radius vector, measured from the same point, of the particle whose motion we wish to determine ; and z the angle between r and p. The satellite moves in an elliptic orbit about the planet, and the acceleration relatively to the planet’s centre of the satellite is (M+m)∕r2 towards the planet along the radius vector r. Now the centre of inertia of the planet and satellite remains fixed in space, and the centre of the planet describes an orbit round that centre of inertia similar to that described by the satellite round the planet, but with linear dimensions reduced in the proportion of m to M+m. Hence the acceleration of the planet’s centre is m∕r2 towards the centre of inertia of the two bodies. Thus, in order to reduce the planet’s centre to rest, we apply to every particle of the system an acceleration m∕r2 parallel to r, and directed from satellite to planet.

Now take a set of rectangular axes fixed in the planet, and let M1r, M2r, M3r be the coordinates of the satellite referred thereto ; and let ξp, ηρ, ζρ be the coordinates of the particle P whose radius is p. Then the component accelerations for reducing the planet’s centre to rest are - mM1∕r2, - mM2∕r2, - mM3∕r2 ; and since these are the differential coefficients with respect to ρξ, ρη, pς of the function ∞p,,ir

“ÿa (Μι£+Μ2Ί+Μ3&

and since cos z=Mj∣ + M2ιj + M3f, it follows that the potential of the forces by which the planet’s centre is to be reduced to rest is

mp

~ cos z.

r2

Now let us consider the other forces acting on the particle. The planet is spheroidal, and therefore does not attract equally in all directions ; but in this investigation we may make abstraction of the ellipticity of the planet and of the ellipticity of the ocean due to the planetary rotation. This, which we set aside, is considered in the theories of gravity and of the figures of planets. Outside of its body, then, the planet contributes forces of which the poten­tial is M/p. Next the direct attraction of the satellite contributes forces of which the potential is the mass of the satellite divided by the distance between the point P and the satellite ; this is—

rn-

√{r2 + p2- 2rpcosz}'

To determine the forces from this potential we regard ρ and z as the variables for differentiation, and we may add to this potential any constant we please. As we are seeking to find the forces which urge P relatively to M, we add such a constant as will make the whole potential at the planet’s centre zero, and thus we take as the potential of the forces due to the attraction of the satellite—

m m

√ {r2+p2 - 2rp cos z} r

It is obvious that r is very large compared with ρ, and we may therefore expand this in powers of p∕r. This expansion gives us

+ + &c.| ,

where P1=cosz, Z2=f cos2z-⅜, P3=f cos3z-⅜ cos z, &c. Tho reader familiar with spherical harmonic analysis of course recog­nizes the Legendre’s functions ; but the result for a few terms, which is all that is necessary, is easily obtainable by simple algebra.

Now, collecting together the various contributions to the potential, and noticing that · cos z, and is therefore equal and oppo­

site to the potential by which the planet’s centre was reduced to rest, we have as the potential of the forces acting on a particle whose coordinates are pξ, ρη, pς j+7^2(icos22-⅜)+7^f(⅛cos32-∣cosz)+ (U-

The first term of (1) is the potential of gravity, and the terms of the series, of which two only are written, constitute the tide-gener­ating potential. In all practical applications this series converges so rapidly that the first term is amply sufficient, and thus we shall generally denote

U=gp2(cos≡z-i) (21

as the tide-generating potential.

In many mathematical works the tide-generating force is pre­sented as being due to an artificial statical system, which produces nearly the same force as the dynamical system considered above. This statical system is as follows. Stopping all the rotations, we divide the satellite into two equal parts, and place them diametri­cally opposite to one another in the orbit. Then it is clear that, instead of the term

m m

√{r3+p2-2rρcosz} r’ we have

⅜rn ⅜ot m √{r2+p2-2rpcosz} + √ {r2+p2+2rp cos z} ∕

And this reduces to

Jp2Λ+>Λ+∙∙∙

The first term is the same as before ; hence the statical system produces approximately the same tide-generating force as the true system. The “moon” and “anti-moon,” however, produce rigor­ously the same force on each side of the planet, whereas the true system only satisfies this condition approximately.@@1

@@@1 The reader may refer to Thomson and Tait’s Natural Philosophy (1883), part ii. §§ 798-821, for further considerations on this and analogous subjects, together with some interesting examples.