§ 6. Form of Equilibrium.

Let us consider the shape assumed by a layer of fluid of density σ, lying on a globe of mass M when acted on by disturbing forces whose potential is Γ=^p2(cos22- ⅛) (3).

Suppose the layer to be very thin, and that the mean radius of the layer is a, and let the equation to the boundary of the fluid be ρ=α[l + e(cos2z- ⅓)] (4).

We assume this form, because the theory of harmonic analysis tells us that the departure from sphericity must be represented by a function of the form cos2 z- ⅓. That theory also gives us as the potential of a layer of matter of depth rα(cos2z- ⅓), and density σ, ut an external point the value

fιrσα2(0 c(cos22-1).

Hence the whole potential, outside of and up to the fluid layer, is 7+I^p2 (cos,s z~ ⅛)+⅜7rσα2(^) e(co≡2 <5)∙

The first term of (5) is the potential of the globe, the second that of the disturbing force, and the third the potential due to departure from sphericity.

Now the fluid must stand in a level surface ; hence, if we equate this potential to a constant, we must get back to the equation (4), which was assumed to be that of the surface. In other words, if we put p = α[l + e(cos2z-⅓)] in (5), the result must be constant, provided the departure from sphericity is small. In effecting the substitution for p, we may put p=a in the small terms, but in the first term of (5) we put

^=⅛-ε(cos2z-⅛)].

p CL

The whole potential (5) can only be constant if, after this substitu­tion, the coefficient of cos2 z — ⅓ vanishes. Thus we must have

M Zma? . n n

-^+-^r+t"Λ=o∙

But if δ be the mean density of the planet M=4/3πα3δ, and gravity g=M∣a2. Then we easily find that

3rnα 1

ι~ 2gr3 l-fyr∣δ w∙

Thus the equation to the surface is

r=α{1+⅞Pi⅛∕j<c°β,≈- w} (7)·

If σ be small compared with δ, the coefficient is 3ma∣2gr3 ; thus we see that 1∕(1 - 3/5σ∕δ) is the coefficient by which the mutual attrac­tion of the fluid augments the deformation of the fluid under the action of the disturbing force. If the density of the fluid be the same as that of the sphere, the augmenting factor becomes 5/2, and we have ε — 15/4ma∣gr2, which gives the form of equilibrium of a fluid sphere under the action of these forces. Since ∣^=^(1 - ∣θε, it follows that, when the form of equilibrium is p—α[l + r(cos2 z - ⅜)], the potential of the forces is

r=!(1-⅛)∙',,<cos,≈-i> <8>∙

More generally, if we neglect the attraction of the fluid on itself, so that σ∕δ is treated as small, and if ρ=α(l + ς) be the equation to the surface of the fluid, where ς is a function of latitude and longi­tude, then the potential of the forces under which this is an equi­librium form is

r=⅛ (9).

a x

It thus appears that we may specify any tide-generating forces by means of the figure of equilibrium which the fluid would assume under them, and in the theory of the tides it has been found prac­tically convenient to specify the forces in this way.

By means of the principle of “ forced vibrations ” referred to in the historical sketch, we shall pass from the equilibrium form to the actual oscillations of the sea.

§ 7. Development of Tide-generating Potential in Terms of Hour-Angle and Declination.

We now proceed to develop the tide-generating potential, and shall of course implicitly (§ 6) determine the equation to the equili­brium figure.

We have already seen that, if 2 be the moon’s zenith distance at the point P on the earth’s surface, whose coordinates referred to A, B, C, axes fixed in the earth, are aξ, aη, aϚ, then cos z=ξM1 + ηM2+Ϛ M3,

where M1, M2, M3 are the moon’s direction cosines referred to the same axes. Then with this value of cos z—

cos2 z - ⅛=2fηMiM3+2⅛ \*⅛2 + 2tfMaMi + 2ξfM1M3

3 ga+yi-2Γ2 M12 + M22-2M32

+ 2 3 3 μ λ

The axis of C is taken as the polar axis, and AB is the equatorial plane, so that the functions of ξ, η, Ϛ are functions of the latitude and longitude of the point P, at which we wish to find the potential·

The functions of M1, M2, M3 depend on the moon’s position, and we shall have occasion to develop them in two different ways,— first in terms of her hour-angle and declination, and secondly (§ 23) in terms of her longitude and the elememts of the orbit.

Now let A be on the equator in the meridian of P, and B 90o east of A on the equator. Then, if M be the moon, the inclination of the plane MC to the plane CA is the moon’s easterly local hour-angle. Let h=local hour-angle of moon and δ=moon’s declination : we have

M1 = cosδcosh, M2 = ∞sδsinh, M3=sin δ, whence

2M1M2=cos2 δ sin 2h, M12 - M22 = cos2 δ cos 2h,

2M2M3=2 sin δ cos δ sin h, 2M1M3=2 sin δ cos δ cos h,

ό

Also, if λ be the latitude of P,

ξ=cos λ, η=0, Ϛ=sin λ, and

⅛=0,-ζp2=⅜cos2λ, er=⅜sin2λ, ^=0, i(ts+√s-2f2)=i-8≡sλ.

Hence (10) becomes

cos2 z - ⅓=⅛ cos2 λ cos2 δ cos Th + sin 2λ sin δ cos δ cos h

+ 3/2(⅓-sin2δ)(⅓-sin2λ) (11).

The angle h, as defined at present, is the eastward local hour-angle, and therefore diminishes with the time. As, however, this function does not change sign with h, it will be more convenient to regard it as the westward local hour-angle. Also, if h0 be the Greenwich westward hour-angle at the moment under consideration; and l be the west longitude of the place of observation P, we have

h=h0-l (12).

Hence we have at the point P, whose radius vector is a,

{⅜ cos2 λ cos2 δ cos 2(λ0 -1) + sin 2λ sin δ cos δ cos (⅛0 - ί)

2z^ +⅜ (⅛-sin2δ)(i-sin2λ)} (13).

The tide-generating forces are found by the rates of variation of V for latitude and longitude, and also for radius a, if we care to find the radial disturbing force.

§ 8. Evaluation of Tide-generating Forces, and Lunar Deflexion of Gravity.

The westward component of the tide-generating force at the earth’s surface, where ρ=a, is dV∣a cos λdl, and the northward component is dV∣adλ ; the change of apparent level is the ratio of these to gravity g. Therefore, differentiating (13), changing signs, and writing ∣^(^ for we have component change of level south­ward

= ⅛(~) {sin 2λ cos2 δ cos 2(A0 - Z) - 2 cos 2λ sin 2δ cos (A0 - Z) 43Λ rf + sin 2λ(l - 3 sin2 δ)} ;

component change of level westward

=s37(-Y {cos λ cos2 δ sin 2(Λ0 - Z)

2M∖rj +sinλsin2δsin(½υ-Z)} (14).

The westward component is made up of two periodic terms, one going through its variations twice and the other once a day. The south­ward component has also two similar terms ; but it has a third term, which does not oscillate about a zero value. If Δ be a de­clination such that the mean value of sin2δ is equal to sin2∆, then, to determine the southward component so that it shall be a truly periodic function, we must subtract from the above sin 2λ(l - 3 sin2 Δ), and the last term then becomes

3 sin 2λ(sin2 Δ - sin2 δ).

In the case of the moon, Δ varies a little according to the position of the moon’s node, but its mean value is about 16° 31'.

The constant portion of the southward component of force has its effect in causing a constant heaping up of the water at the equator ; or, in other words, the moon’s attraction has the effect of causing a small permanent ellipticity of the earth’s mean figure. This augmentation of ellipticity is of course very small, but it is necessary to mention it in order that the meaning to be attributed to lunar deflexion of gravity may be clearly defined.

If we consider the motion of a pendulum-bob during any one day, we see that, in consequence of the semi-diurnal changes of level, it twice describes an ellipse with major axis east and west, with ratio of axes equal to the sine of the latitude, and with linear dimensions proportional to cos2 δ, and it once describes an ellipse whose north and south axis is proportional to sin 2δ cos 2λ and whose east and