⅛-+⅛β⅛w=to^+τ⅛=0 <19>∙

§ 12. Adaptation to Forced Oscillations.

Since we may suppose that the free oscillations are annulled by friction, the solution required is that corresponding to forced oscil­lations. Now we have seen from (13) that ε (which is proportional to V) has terms of three kinds, the first depending on twice the moon’s (or sun’s) hour-angle, the second on the hour-angle, and the third independent thereof. The coefficients of the first and second terms vary slowly, and the whole of the third varies slowly. Hence t has a semi-diurnal, a diurnal, and a long period term. We shall see later that these terms may be expanded in a series of approxi­mately semi-diurnal, diurnal, and slowly varying terms, each of which is a strictly harmonic function of the time. Thus we may assume for t a form e cos (2nft + kφ + a), where f and k are numbers, and where e is only a function of co-latitude and of the elements of the orbit of the disturbing body. According to the usual method of treating oscillating systems, we may therefore make the follow­ing assumption for the form of solution

t =e cos (2n∕J + fc≠ + α) 1

fj = hcos(2n∕ι! + ⅛≠ + a) ( ∕20∖

ξ = xcοs(2nft + kφ + a) l ’ '"' h

η = y sin (2nft + kφ + a) '

where e, h, x, y are functions of co-latitude θ only. Substituting these values in (19), we have

s⅛⅛<1'xsin9> + tw + hα = 0 (2I)·

Then, if we write u for h - e, and put m=n2a∣g, substitution from (20) in (17) leads at once to

x√.+y∕staβcos.=i⅛ I w

y∕.8iα.+V∞.\*=-⅛ri‰∫

Solving (22) for x and y, we have

, μ . λ. 1 cdu k cos 0∖ λ \*',-",'⅛(δ+7⅛) I ™3) ysin^-eos≈<,) = ∕⅛⅛ + ⅛)∣ j v ' ½rn∖ f dθ sιnθj)

Then substituting from (23) in (21), we have

, , cosØ du fru

1 d Γ√sin0⅛n + > ucos0∏ \*∕ c?0 sinØ

⅛π⅞L ~~∕-oo<m~~—'J^⅛-<v2~ ∞≡-∙")'

+ 4τnα(u + e)=0 (24).

This is Laplace’s equation for tidal oscillations in an ocean whose depth is only a function of latitude. When u is found from this equation, its value substituted in (23) will give x and y.

§ 13. Preparation for Solution.

The ocean which is considered in this case is not like that on the earth’s surface, and therefore it does not seem desirable to pursue the integration of (24) except in certain typical cases.

In (13) we have the expansion of the disturbing potential and implicitly of the disturbing forces in three terms, the first of which is variable in half a day, the second in a day, and the third in half the period of revolution of the tide-raising body. Forestal­ling the results of chapter iv.—each of these terms may be expressed as the sum of a series of strictly harmonic functions of the time ; the first set of these have all approximately semi-diurnal periods, the second approximately diurnal periods, and the third vary slowly in dependence on the periodic time of the tide-generating body. The first set involve twice the terrestrial longitude, the second the longitude, and the third set are independent of the longitude of the place of observation. From these statements compared with (13) we see that in the semi-diurnal terms f is approximately unity, k=2, and e=E sin2 θ ; in the diurnal terms f is approximately ½, k=1, and ε = E sin θ cos θ ; in the terms of long period f is a small fraction (for the fortnightly tide about 1/28), k=0, e=E(⅓- cos2 θ). The departure from exactness in the rela­tion f=1 for the semi-diurnal, and f=½ for the diurnal terms is generally (except for certain critical depths of ocean) not such as to greatly change the nature of the results from those obtained when f=1 and ½ rigorously. Hence the integration of (24) will be pursued on these three hypotheses, giving Laplace’s three kinds of oscillation. The hypothesis which wall be made with regard to is that γ = l(1 - q cos2 θ), and in the case of the semi-diurnal tides we shall be compelled by mathematical difficulties to suppose q to be either unity or zero. The tides of zonal seas may be worked out, and more complex laws of depth may be assumed ; but for the discussion of such cases the reader is referred to Thomson’s papers in Phil. Mag., 1875.

There might be reason to conjecture that the form of u would be similar to that of e, and this is in fact the case for the diurnal tides for any value of q and for the semi-diurnal tides when q is unity. Before proceeding further it will be convenient to exhibit two purely analytical transformations of the first two terms of (24) which hold true for certain values of k and f and when u has such a form as that suggested. If we put k=1, f=½, γ=l(1-q cos2 θ), then, if v=A sin θ cos θ, it will be found on substitution that

, γ(sin 0^ + 2vcosø) 2cot0^ + =Λ=

1 d '∖ dθ ) dθ sιn20 ο1 ,oκ.

*SiM*~~v~~~~~ ⅜-co~~~~s~~~~^ = ^~~~~8~~~~⅛\* -<~~~~25,~~~~∙~~

Again, if we put k=2, f=1, q = 1, γ=l(1 - cos2 θ)=l sin20, and if v=A sin2 θ,

, 7(sinfl^ + 20cos0)

1 α '∖ dθ ∕ o dθ sιn20 o,

sin0(Z0 l-cos20 2γ l-cos20 -

Another general property of (24) is derived from the supposition that u is expressed in a senes proceeding by powers of l ; thus u=v0+v1^ + r2 + (27).

Let v0, υ1, v2, &c., be so chosen that, when u is substituted in (24), the coefficient of each power of l vanishes independently ; then the term independent of l obviously gives v0= -e, and the connex­ion between successive v’s is

1\_ c7 I 7 ( sin θ~dJ + 7r\*∙cos θ) \_ kS f dθ+sinθ)

sin θ dθ /2 \_ c^~~os~~~~2~~ ~~θ~~ J 7 sin 0 (∕2 - cos2 0)

+ 4τ⅛+1 = 0 (28).

We shall suppose below that u is expansible in the form.(27), and shall use (28) in conjunction with (25) or (26) for finding the successive values of the v’s.

§ 14. Diurnal Tide.

Let us first consider the diurnal tides. We have e = E sin θ cos θ, k= 1, and f=½ ; then v0= - E sin θ cos θ. Hence by (28) and (25) -⅛lqv0 + 4mh>1 = Q (29),

and therefore », = — v0, Applying the same theorem a second time, »2=(⅝lm)vv an4 sθ on ; therefore u = »θ[1 + 2lq∕ma + (2⅛∕∞α)2+...]

\_ ι⅛ \_ \_ e <afn

1 - 2lq∣ma 1 - 2lq∣rna ' '"

Butu=h-ej hence . 2lq∣ma

h=-Γ⅛Se∙ ■■■; <31>∙

It appears, therefore, that the tide is “ inverted,” giving low water where the equilibrium tide gives high water. If q=0, so that the ocean is of uniform depth, the tide vanishes.

§ 15. Semi-Diurnal Tide, with Variable Depth.

Next let us consider the semi-diurnal tide in the case where q=1, so that y=l sin2 θ. Then e = E sin2 θ, k=2, f=1 ; also v0= — e= - E sin2 θ. Hence by (28) and (26) - 8lv0 + 4mlv1 = 0, whence v1 = 2∕mv0. Applying the same theorem a second time, v2=(2∕m)2v0, and so on ; therefore u=v0[1 + 2l∣ma + (2l∕ma)2+...]

\_\_ t⅛ \_ e

1 - 2l∣ma 1 - 2l∣ma^

Hence h=u+e= - ~~,~~ ~~2ι~~~~^e~~ (32).

If 2l∕ma= ½, the height of tide is equal to the equilibrium height ; but it is inverted, giving low water where the equilibrium theory gives high water. In the case of the earth m=1/289, and therefore this relation is satisfied if l=a∕1156. Hence in a sea 3000 fathoms deep at the equator, and shallowing to the poles, we have inverted semi-diurnal tides of the equilibrium height.

§ 16. Semi-Diurnal Tide, with Uniform Depth.

The method of development used above, where we proceed by powers of the depth of the ocean, is not applicable where the depth is uniform, because it leads to a divergent series. We have there­fore to resume equation (24). In the case of the semi-diurnal tides we have for the depth y=l (a constant by hypothesis), k=2, f=1 approximately, and e=Esin2θ. Now for brevity let β=4ma∣l, v = sin θ, so that e = Ev2. Then we find that on development (24) becomes

f≡(1 \_ vrfC± - - (8 - 2p2 \_ ∕¾3)u = - ∕3Ep6 (33).

Let us now assume as the solution of this equation

u=(K2 - E>2+Kivi + Kβvβ +...+ K2iV-i+ (34).

Substituting from (34) in (33), and equating to zero the coefficients of the successive powers of v, we find K2=E, K4 apparently inde­terminate, and

2i( 2i + 6 ) K2i+4 — 2i(2i + 3) K2i+2+βK2i= 0 (35).

Since K0=0, this equation of condition may be held to apply for all positive integral values of i, beginning with i=0. It is obvious that K6 is determinable in terms of K4 and K2, K8 in terms of K6