and K4, &c., so that all the K's are to be found in terms of K2, which is known, and of K4, which is apparently indeterminate.

The condition for the convergency of the series (34) for u and for the series du∣dv is that K2i+2∣K2i shall tend to a limit less than unity. The equation (35) may be written

∙^≈+<-2i+3 β K\*

‰2 2i + 6 2ι(2ι + 6) Éü+2 (36).

Now K2i+2∣ K2i tends to be either infinitely small or not infinitely small. If it be not infinitely small in the limit, the second term on the right of (36) becomes evanescent when i is very great, and we have in the limit when ί is very large—

2f±\_3 Γ 3Ί Γ \_ 3-1

‰2κ2i+2 2ι + 6 L 2(ι+ 3) J L 2Ϊ J

But the ratio of successive terms of √(1 - v2) tends to become (1 - 3/2/i)v2. Hence, if K2i+2∕K2i does not tend to become infinitely small, u=A + B√l -v2, where A and B are finite for all values of v. Again, under the same circumstances we have in the limit when i is very large—

(2i + 4)A2i44t>2^ \_2i + 4 2i + 32Λ , 1 ∖Λ 8\_\ 2

(2i + 2)K2^iVi+i~2i+2'2i+6 ~V 1 + 1Λ 2(ι+3)Λ

=(l-l∕2t>2.

But the ratio of successive terms of (1 — v2)-½ tends to (1 - ½/i)v2. Hence, if K2i+i/K2i does not tend to become infinitely small, du∕dv=C + D(1 -v2)-½, where C and D are finite for all values of v. Now ⅛=¾^ V1-f2=C√1 →2+D.

dθ dv

Therefore at the equator, where v=1, du/dθ = D, a finite quantity. Hence the hypothesis that K2i+2∣K2i tends to be not infinitely small leads to the conclusion that u and dn∕dθ are finite at the equator. But on account of the symmetry of the system the co-latitudinal displacement ξ must vanish at the equator, and therefore x also. By (23), when f=1, k=2, v=sin θ,

a-≡(5+8∙m∙\*)∙

But we have just seen that this hypothesis makes u finite when v = 1 or θ=90°, and therefore at the equator

1 rfα c ∙a ...

Χ=4»ϊ dθ, a finite quantity.

Now symmetry necessitates a vanishing value of du∣dθ at the equator. Thus the hypothesis that K2i+2∣K2i tends to be not in­finitely small is negatived, and we conclude that, on account of the symmetry of the motion, it is infinitely small for infinitely great values of i. This being established, let us write (36) in the form

∙ga+⅞- IÉ (36a).

AΓ2j 2i2+3 ί — ( 2i2 + 6i) K3t+4∣ K2i+2

Hence by repeated application of (36a) we have

⅜β , ,, , o,∙xo

~~a~~~~'≈~~~~,~~ ~~~~~~~2Î~~~~\*~~~~+M~~~~-ai ∣ ιri⅛ 1 n~~ t(<+i),+⅜<+i)P

2(⅛ + I) +3(ι + l) 2(i+2)2 + 3(i + 2)-⅛c. (37). And we know that this is a continuous approximation to K2i+2∕K2i, which must hold in order that the latitudinal velocity may vanish at the equator. Writing Ni= K2i+2∕ K2i∙, all the N’s may be com­puted from the continued fraction (37). Then

K2=E, K4∕E = N1, K6∕E = N1N2, K8/E = N1N2N3 &c.

We cannot compute K6 from K4, K8 from K6, and so on ; for, if we do, then, short of infinite accuracy in the numerical values, we shall be gradually led to successive values of the K’s which tend to equality.@@1

This process was followed by Laplace without explanation. It was attacked by Airy in his “ Tides and Waves ” (in Ency. Metrop.) and by Ferrel in his Tidal Researches (U.S. Coast Survey, 1873), but was justified by Sir W. Thomson in the Phil. Mag. (1875, p. 230). The investigation given here is substantially Thomson’s.

Laplace gives numerical solutions for three different depths of the sea, 1/2890, 1/722⋅5, 1/361⋅25of the earth’s radius. Since m=1/289, these correspond respectively to the cases of β=40, 10, 5, and the solutions are

β=40, h=E{v2+20⋅1862v4 +10∙1164v6 - 13⋅1047v8 - 15∙4488v10 -7⋅4581v12- 2∙1975v14- 0∙4501vl6 - 0∙0687v18 -0∙0082v2°- 0⋅0008v22- 0⋅0001v24.. .}

β=10, h = E{v2 + 6⋅1960v4 + 3∙2474v6 + 0⋅7238v8+ 0⋅0919v1°

+ 0O076v12+ 0O004p14...}

/3= 5, h = E{χ2+ 0,7504^4 + 0∙1566p6 + OO157>'8 + OOOO9i'lo + ...}

Since h vanishes when v = 0, there is no rise and fall of water at the poles. When v = 1 at the equator, we find

β=40, h= -7∙434E β=10, h= 11∙267E β= 5, h= 1⋅924E.

The negative sign in the first case shows that the tide is inverted at the equator, giving low water when the disturbing body is on the meridian. Near the pole, however, that is, for small values of v, the tides are direct. In latitude 18° (approximately) there is a nodal line of evanescent semi-diurnal tide. In the second and third cases the tides are everywhere direct, increasing in magnitude from pole to equator. As β diminishes the tides tend to assume their equilibrium value, because all the terms, save the first, become evanescent. When β=1 (depth 1/72 of radius) the tide at the equator still exceeds its equilibrium value by 11 per cent As β diminishes from 40 to 10 the nodal line of evanescent tide contracts round the pole, and when it is infinitely small the tides are infinitely great. The particular value of β for which this occurs is that where the free oscillation of the ocean has the same period as the forced oscilla­tion. The values chosen by Laplace were not well adapted for the illustration of the results, because in the cases of β=40 and β = 10 the depth of the ocean is not much different from that value whieh would give infinite semi-diurnal tide. For values of β greater than 40 we should find other nodal lines dividing the sphere into regions of direct and inverted tides. We refer the reader to Sir W. Thomson’s papers for further details on this interesting point.

§ 17. Tides of Long Period; Laplace’s Argument

from Friction.

In treating these oscillations Laplace remarks that a very small amount of friction will be sufficient to cause the surface of the ocean to assume at each instant its form of equilibrium, and he adduces in proof of his conclusion the considerations given below. The friction here contemplated is such that the integral effect is represented by a retarding force proportional to the velocity of the water relatively to tho bottom. Although proportionality to the square of the velocity would probably be nearer to the truth, yet Laplace’s hypothesis suffices for the present discussion.

In oscillations of this class the water moves for half a period north, and then for half a period south. In oscillating systems, where the resistances are proportional to the velocities, it is usual to specify the resistance by a modulus of decay, namely, that period in which a velocity is reduced to e-1 of its initial value by friction. Now the friction contemplated by Laplace is such that the modulus of decay is short compared with the semi-period of oscillation. The quickest of the important tides of long period is the fortnightly (see chapter iv.) ; hence, for the applicability of Laplace’s conclusion, the modulus of decay must be short compared with a week. Now it seems prac­tically certain that the friction of the bed of the ocean would not materially affect the velocity of a slow ocean current in a day or two. Hence we cannot accept Laplace’s discussion as satisfactory. How­ever this may be, we now give what is substantially his argument.

Let us write ϐ for the reciprocal of the modulus of decay. Then the frictional forces introduced on the left-hand side of (17) are + ϐdξ∣dt in the first and sin θϐdη∣dt in the second. Laplace’s hypothesis with regard to the magnitude of the frictional forces enables us to neglect the terms d2ξ∣dt2 and sin θ d2η∣dt2 compared with the frictional forces. Then, if we observe that in oscillations of this class the motion is entirely latitudinal, equations (17) and (19) become fξ n . a adη g d ,, . y

- 2π sin 0 cos 0 √ = - j√h - r) dt dt ad<Γj 1

sin 00^+2n cos 0^=0 ► (38).

⅛α sin 0 + ^(γ⅜ sin 0) = 0

From the first two of these we easily obtain 0+‰a¾-≈-⅛⅛-t> (39).

As a first approximation we treat dξ∣dt as zero, and obtain h=e, or the height of water satisfies the equilibrium theory. In these tides (see chap. iv.) f = E (⅓-cos2θ) cos it, so that from the third equation of (38) we can obtain a first approximation to ξ; then, sub­stituting in (39), we obtain on integration a second approximation to h. Laplace, however, considers as adequate the first approxima­tion, which is simply the conclusion of the equilibrium theory.

§ 18. Tides of Long Period in an Ocean of Uniform Depth.

As it seems certain that these tides do not satisfy even approxi­mately the equilibrium law, we now proceed to find the solution where there is no friction. In the case of these tides k=0, f a small fraction, and e=E (⅓-cos2θ). The equation (24) then becomes ii⅛⅛fc⅛)+4≡ <"+β>=0 ■·

y2 - cos20,

@@@1 Thomson calls this a dissipation of accuracy. It may be illustrated thus. Consider the equation *x*2-3*x*+2=0, which may be written either *x*=⅔+⅓*x*2 or *x=3-2x.* Now let *xn*+1=⅔+⅓*x*2n, and suppose we start with any value *x*0, less than unity, and compute *x*1, *x*2,... *xn.* Then, starting with *xn* in the equation xn-1=3-2∕*xn*, if we work backwards, we ought to come to the original value *x*0. In fact, however, we shall only do so if there is infinite accuracy in all the numerical values. For, start with *x*0=½, then *x*1 = ⋅75, *x*2= ⋅8542, *x3=* ∙9099, *x4=* ∙9527, *x5=* ⋅9692 ; and the values go on approximating to 1, which is a root of the equation. Next start backwards with *x5=* ∙97, and we find *x4=* ∙938, *x*3= ∙868, *x2 =* ∙696, *x*1=∙127, *x*0= -12∙75, x-1=3·157, x-2 = 2⋅367, x-3=2∙155, x-*x*-4=2⋅072 ; and the values go on approximating to 2, the other root of the equation.