or, writing μ for cos θ and e=E (⅓ - μ2),

(40)·

We shall confine the investigation to the case where γ = l, a con­stant, and where the sea covers the whole surface of the globe. The symmetry of the motion in this case demands that u when ex­panded in a series of powers of μ shall only involve even powers. Let us assume, therefore, that

^Z7s⅛=biM+b3^∙+- ÷b≡H-^"+1+ (\*1)∙

τheu ⅛≡72 s=B1/x+(B=» " B1>s + ∙∙ · + <b\*+1 - b≡∙-i>\*+1+· ∙∙

⅛(⅛^∣)=b>+s<b>-b>>,+∙∙∙

+ (2ι +1) (Ba+ι - B2i-ι)μ24 + (42).

Again,

g= -∕2B1μ+(B1 -∕2B3>3 + ... + (B2i-ι -∕‰+1>2i+1+ (43),

u=C-irB1μ2 + 4(B1-√2B3>\*+ ... +^(B2i-3-∕≡B24-1>21+ (44), where C is a constant. Then, writing β for 4mα∣l, as in the case of the semi-diurnal tide, substituting from (42), (43), and (44) in (40), and equating to zero the successive coefficients of the powers of μ, we find C = - ⅓E + B1∕β>

B1-B1(l-⅛∕Ι3) + ⅜∕3E = 0

V 2∙3 7 4(45).

Ba+1 - Ba-ι(l - 2i(2i + 1) PP ) - 2i(2^+l) δ2i-3=0

Thus the constant C and B3, B5, &c., are all expressible in terms of B1, and B1 is apparently indeterminate. We may remark that, if -⅛~1=^E, or B-1=-2E,

the equation of condition (45) may be held to apply for all values of i, from one to infinity. Let us write (45) in the form

⅛±1 \_ 1 \_ \_\_1 ∕2fl X ß B2i-3

Ba-ι 2τ(2i+l)7p+ 2i(2i+l)Ba-1 l λ

When i is large B2i+1/B2i-1 either tends to become infinitely small or it does not do so. Let us suppose that it does not tend to become infinitely small. Then it is obvious that the successive B’s tend to become equal to one another, and so also do the values of (B2i-2-f2B2i-1) ∕2i and the coefficients of du/dμ. Hence we have du/dμ = L + M∣(1- -μ2), for all values of μ, where B and M are finite. Hence this hypothesis gives infinite velocity to the fluid at the pole, where μ=1. But with a water-covered globe this infinite velocity is impossible, and therefore the hypothesis is negatived, and B2i+1∕B2i-1 must tend to become infinitely small. This being established, let us write (46) in the form

β

B2i-i\_ 2ι(2t + l)

B2i,3~ Pβ B2,∙+1 v λ

2i(2i + l) Ba-ι

By repeated applications of (47), we have in the form of a con­tinued fraction

β

B2i-ι~ 2i(2i+l) β

B j-3 ^ 1 \_ W + (23+2j(2i+3) β

2ι(2i + l) 1 f-β (2i+4)(2i+5)

(2i+2)(2i+3) + - Pβ ι

1 (2i+4)(2i + 5) + fe c∙(48)∙ And we know that this is a continuous approximation, which must hold in order to satisfy the condition that the water covers the whole globe. Let us denote this continued fraction by - Ni. Then, if we remember that B-1= - 2E, we have

B1=2EN1, B3∕B1= -N2, B5∕B3= - N3, B7∕B5= - N4, &c., so that

B3= -2EN1N2, B5=2EN1N2N3, B7= - 2EN1N2N3N4, &c., and C=-⅓E+2EN1∕β.

Then h = u + e

= C + ⅜E-(E +V-B1)μ2 + ⅜(B1-∕2B3)√ + ⅛(B3-∕2B5)Mβ + ... =E {2N1∣β - (1 +∕2A'1)∕λ2 + ⅛W1(1 +f2N2)μ\*

-⅛W1A2(l+∕∙W3>≡+...} (49).

Now we find that, when β=40, which makes the depth of the sea 3000 fathoms or 1/2890 of the radius of the earth, and with f=⋅0365012, which is the value for the fortnightly tide (see chap. iv.),

N1 = 3∙040692, N2=1∙20137, N3 = ∙66744, N4 = ∙42819, N5=∙29819, N6=∙21950, N7=∙16814, N8=∙13287, N9=∙107, N10=∙1.

These values give

2W√∕3=∙15203, 1+∕2W1 = 1OO41, ⅛N1(1 +∕W2) = 1'5228, ⅛W1W2(1 +∕2W3) = 1∙2187, ⅛W1W2A^3(1 +∕2A4) = ∙60988, ⅜W1... W4(l +∕W5) = ∙20888, IN1... Λτ5(l +∕2W6) = ∙05190, jN1... Λrβ(l +f2N7) = ∙00976, ⅛ N1... W7(l +∕W8) = \*0014, ⅜W1∙.∙A-8(1+∕2Λ79) = ∙00017.

So that

h∕e= {∙1520 - lO041μa + l∙5228μ4 - Γ2187μ6+∙6099μ8 - ∙2089μw + O519√3 - ∙0098μ1\* + -0014μ16 - ∙0002μ18} ÷(⅛ - μ2) (50). At the pole, where μ=1, h= -E x ⋅1037=e × ∙15561 and at the equator, where μ,=0, h=+E× ∙1520 = e × ,4561 ∫ ' '-

Now let us take a second case, where β = 10, which was also one of those solved for the case of the semi-diurnal tide by Laplace, and we find

h∕E= ∙2363-1-0016/?+ '5910/?- ‘1627/?+ ,0258μ8- ∙0026μ1°

+ ∙0002√2.

At the pole, where μ=l, we find h = - Ex ⋅3137 = ⋅ ⋅471, and at the equator h=+E× ⋅2363 = e x ⋅709. With a deeper ocean we should soon arrive at the equilibrium value for the tide, for N1, N2, &c., become very small, and 2N1∕β becomes equal to ⅓. In this case, with such oceans as those with which we have to deal, the tides of long period are considerably smaller than the equilibrium value.

§ 19. Stability of the Ocean.

Imagine a globe of density δ, surrounded by a spherical layer of water of density σ. Then, still maintaining the spherical figure, and with water still covering the nucleus, let the layer be displaced sideways. The force on any part of the water distant r, from the centre of the water and r from the centre of the nucleus is 4/3πσr, towards the centre of the fluid sphere and 4/3π(δ - σ)r towards the centre of the nucleus. If δ be greater than σ there is a force tend­ing to carry the water from places where it is deeper to places where it is shallower ; and therefore the equilibrium, thus arbitrarily dis­turbed, is stable. If, however, δ is less than σ (or the nucleus lighter than water) the force is such that it tends to carry the water from where it is shallower to where it is deeper, and therefore the equilibrium of a layer of fluid distributed over a nucleus lighter than itself is unstable. As Sir William Thomson has remarked,@@1 if the nucleus is lighter than the ocean, it will float in the ocean with part of its surface dry. Suppose, again, that the fluid layer be disturbed, so that its equation is r=a(1 +si), where si is a sur­face harmonic of degree i ; then the potential due to this deforma­tion is ~πσ^ ⅛-τ si, and the whole potential is

2ι + l r\*+l ’ r

4∏∙δα3 4τr<r «ί+3 3r ^^2ι + l ri+ls\*'

If, therefore, σ∕(2i + l) is greater than ⅓δ, the potential of the forces due to deformation is greater than that due to the nucleus. But we have seen that a deformation tends to increase itself by mutual attraction, and therefore the forces are such as to increase the deformation. If, therefore, σ=⅓(2i + 1)δ, all the deformations up to the ith are unstable, but the i + 1th is stable.@@2 If, however, σ be less than δ, then all the deformations of any order are such that there are positive forces of restitution. For our present purpose it suffices that this equilibrium is stable when the fluid is lighter than the nucleus.

§ 20. Precession and Nutation.

Suppose we have a planet covered with a shallow ocean, and that the ocean is set into oscillation. Then, if there are no external dis­turbing forces, so that the oscillations are “free,” not “forced,” the resultant moment of momentum of the planet and ocean remains constant. And, since each particle of the ocean executes periodic oscillations about a mean position, it follows that the oscillation of the ocean imparts to the solid earth oscillations such that the re­sultant moment of momentum of the whole system remains constant. But the mass of the ocean being very small compared with that of the planet, the component angular velocities of the planet necessary to counterbalance the moment of momentum of the oscillations of the sea are very small compared with the component angular velocities of the sea, and therefore the disturbance of planetary rotation due to oceanic reaction is negligible. If now an external disturbing force, such as that of the moon, acts on the system, the resultant moment of momentum of sea and earth is unaffected by the interaction between them, and the processional and nutational couples are the same as if sea and earth were rigidly connected together. Therefore the additions to these couples on account of tidal oscillation are the couples due to the attraction of the moon on the excess or deficiency of water above or below mean sea-level. The tidal oscillations are very small in height compared with the equatorial protuberance of the earth, and the density of water is 4/11ths of that of surface rock ; hence the additional couples are very small compared with the couples due to the moon’s action on the

@@@1 Thomson and Tait, *Nat. Phil.,* § 816.

@@@2 Compare an important paper by Poincaré, in *Acta Math.* (1885), 7 ; 3, 4.