solid equatorial protuberance. Therefore precession and nutation take place sensibly as though the sea were congealed in its mean position. If the ocean be regarded as frictionless, the principles of energy show us that these insensible additional couples must be periodic in time, and thus the corrections to nutation must consist of semi-diurnal, diurnal, and fortnightly nutations of absolutely insensible magnitude. We shall have much to say below on the results of the introduction of friction into the conception of tidal oscillations as a branch of speculative astronomy.

§ 21. Some Phenomena of Tides in Rivers.

In § 2 we have given a description of some of the phenomena of the tide-wave in rivers. As a considerable part of our practical knowledge of tides is derived from observations in estuaries and rivers, we give an investigation of two of the most important features of the tide-wave in these cases. It must be premised that when the profile of a wave does not present the simple harmonic form it is convenient to analyse its shape into a series of partial waves superposed on a fundamental wave ; and generally the prin­ciple of harmonic analysis is adopted, in which the actual wave is regarded as the sum of a number of simple harmonic waves.

The tide-wave in a river is a “long” wave in which the vertical motion of the water is very small compared with the horizontal, the river very shallow compared with the wave-length, and the water which is at any moment in a vertical plane always remains so throughout the oscillation.

Suppose that the water is contained in a straight and shallow canal of uniform depth ; then take an origin of coordinates at the bottom, with the x axis horizontal in the direction of the canal, and the y axis vertical ; let h be the undisturbed depth of water ; let h + η be the ordinate of the surface corresponding to that fluid whose undisturbed abscissa is x and disturbed abscissa x + ξ ; and let g be gravity. The equations of motion and continuity@@1 are

cPξ x cFξ∕dx2 dfl-ffn(i+^∕dxγ[ -hdlfdx [ (52)∙

η l+dξ∕dx J

For brevity we shall write ν2=gh and u = vt - x. Since for “ long ” waves dξ∣dx is small, the equations (52) become approximately

dcl dxi∖ dx) [ KJLB∖' f

h~ dx'∖dx) J

For finding a first approximation we neglect the second term on the right of each of (53). The solution is obviously ξ=acοsm{vt-x)=acοsmu

∙η= - mah sin mu J '0 h

(54) gives the height of the water whose undisturbed abscissa is x, and since ξ is small this is approximately the height at the point on the bank whose abscissa is x. But now suppose that at the origin (the mouth of the river) the canal communicates with a basin in which there is a forced oscillation of water-level given by η=Hsinnt (55).

the over-tide is 1/10th of the fundamental and has a range of 2 feet. If the river shallows very gradually, the formula will still hold, and we see that the height of over-tide varies as (depth)-3/2.

Fig. 1@@2 read from left to right exhibits the progressive change of shape. The steepness of the advancing crest shows that it is a shorter time from low to high water than vice versa. The law of the ebb and flow of currents mentioned in § 2 may also be easily determined from the above investigation. We leave the reader to determine the effect of friction, which is given by inserting a term - μdξ∣dt on the right-hand side of (57).

(ii.) Compound Tides (see § 24).—We shall now consider the mutual influence of two waves of different periods travelling up the river together. In the first approximation they are quite inde­pendent, and we may assume

ξ=a cos m(vt -x) + b cos [n(vt - x) + ϵ] (61).

In proceeding to the second approximation, we only take notice of those terms which result from the interaction of the two, and omit all others, writing for the sake of brevity

{m-n,i = (m-n)(νt-x) - ϵ,

{m+n∖ = lm + n)(vt -x)+e.

With the value of ξ assumed in (61), we find, on substituting in (53) and only retaining terms depending on mutual influence, that the equations for the second approximation are

This represents the oceanic tide, and n is that which we call below (§ 23) the speed of the tide. Then obviously m=n∣v, so that at any point x up the river

,=Zf3inn(i-^) w

(56) gives the first approximation to the forced tide-wave, and it is clear that any number of oscillations may be propagated inde­pendently up the river with the velocity √gh due to the depth of the river. In passing to the second approximation we must separate the investigation into two branches.

(i.) Over-Tides (see § 24).—We now suppose that the tide at the river mouth is simply (55). On substituting the approximate values (54) in (53) our equations become

\_ dt (57).

h= ~dx ⅛m,2a\* ~ im2di cos 2τnw I

We have now to assume an appropriate form for the solution of (57), such as ξ=a cos mu + Ax cos 2mu + B sin 2mu (58).

We have here in effect assumed that the second and third terms of (58) are small compared with the first. It is clear, however, that at a distance from the origin the term in A will become large. This difficulty may be eluded by taking the canal of finite length, and supposing that, where the canal debouches into a second basin, a second appropriate forced oscillation is maintained. The length of the canal remains arbitrary, save that the second term of (58) shall still be small compared with the first. On substituting from (58) in (57) we have B indeterminate and A= -3/8a2m2 ; hence η∣h = ½m2a2-ma sin mu +¾m3a2xsin2mu + (2mB - ⅛m2a2)cos 2mu (59). This gives the elevation of the water whose undisturbed abscissa is x, that is to say, at the point whose abscissa along the bank is X=χ+ξ. If we put x=X-ξ in the largest term of (59), and treat ξ as small, and put x=X in the small terms, (59) becomes η∕h = - ma sin m(vt - X) + ¾m3a2X sin 2m(νt - X)

+ (2mB - ⅝m2α2) cos 2m(vt - X).

But at the origin (55) holds true, therefore B = 5/16ma2, -mah=H, and mv=n. Thus the solution is

,=B⅛> »(<- ^) + sta \*(<-⅛) (60).

From (60) we can see what the proper forced oscillation at the further end of the canal must be ; but this matter has no present interest. The first term of (60) being called the fundamental, the second gives what is called the first over-tide ; and by further approximation we can get the second, third, &c. The over-tide travels up the river at the same rate as the fundamental, but it has double frequency or “speed,” and the ratio of its amplitude to that of the fundamental is

M ∖jgh

As a numerical example, let the range of tide at the river mouth be 20 feet and the depth of river 50 feet. The “speed” of the semi-diurnal tide is about 1/1⋅9 radians per hour; \*fgh=27 miles per hour ; hence ‰ W°^ss⅞42∙^' Therefore 34 miles up the river

iT-t "∖

=Ι>2^ + iv2αfrmn[(7n + n)sin{m + n} - (m-π)sin{wι-π}] Lθ2 y∣h = - abmn∖cos {m + n} - cos {τn - n} ] - dξ∣dx J

Now let us assume as the solution ξ=acοsm(vt-x) + Axcοs{m+n} +Bsin{m + n} 1 ,g„, .

+ bcos[n(vt-x) + ϵ] + Cxcos{m-n} +Dsin{m-n} J " '∙ ' ’

and let us elude the difficulty about the increasing magnitude of the second term in the same way as before. Substituting in the equation of motion, we have for all time,

2(m + n)A sin {m+n} + 2(m - n)C sin {m - n}

+ 3/2abmn[(m + n) sin {m+n} - {m - n) sin {m - n} ] = 0.

This gives A = - ¾abmn and C = + ¾abmn. B and D remain arbi­trary as before, and will be dropped, because they are to be deter­mined by the condition that at the origin the terms of dξ∣dx in cos {m + n}, cos{m-n} are to vanish, whence

η∣h = - am sin m(vt -x)-bn sin [n(vt - x) + ϵ]

+ ¾abmn[(m+n)x sin {m + n} - (m - n)x sin {m - n} ]

+ terms in cos {m + n} and cos {m-n}.

Then we pass from x to X as in the last section, and make the terms in cos {m + n} and cos {m — n} vanish by proper values of B and D, and we have

η = amhsinm(vt -X)- bnhsin[n(υt - x) + ϵ]

+ ¾abmnX[(m + n) sin {m+n} -(m- n) sin {m - n} ] (64). Now at the river’s mouth, where x=0, suppose that the oceanic tide is represented by η = H, sin n1t +H2sin (n2t + ϵ).

@@@1 See, for example, Lamb's *Hydrodynamics,* chap. vii.

@@@2 From Airy. “Tides and Waves.”