Then -am=H1/h, - bn = H2∕h, habmn=H1H2/h,

w1⅛n2 rnv=nl, nv=n2, v=\*fgh, m±n=

so that (64) becomes

,=¾⅛n1(i--^--) + ¾sin[¾(i-^) + e] +¾-i,√⅛-rsi"B,,ι+^('-⅞)+'. - 34r si', [<“■ - ”«>(' - ⅞) -]∙∙∙ <δ5>∙ As a numerical example, suppose at the mouth of a river 50 feet deep that the solar semi-diurnal tide has a range 2H1 = 4 feet, and the lunar 2H2 = 12 feet ; then n1 + n2=59/57 radians per hour, n1 - n2=1/57 radians per hour, and as before √gh=27 miles per hour. With these figures 3H1H2 n1+n2 1

th Λfgh 170

Thus 15 miles up the river the quater-diurnal tide (in § 24 below, called MS) has a semi-range of an inch. But the luni-solar fort­nightly tide (called MSf in § 24) would have a semi-range of 1/60th of an inch. Where the two interacting tides are of nearly equal speed, the summation compound tide is very large compared with the differential tide. As before, when the river shallows gradually this formula will still hold.

It is interesting to note the kind of effect produced by these compound tides. When the primary tides are in the same phase n1t=n2t+ ϵ.

∖ jr

*t~Mgh) + ~ (nι+nz)~if^i ’*

<n\*-,⅛>(i-⅞)-e= -⅛-"≈>⅞1 and

,=(V1 + ¾)8in≈1(i-^) + ¾S,⅛^⅛ta[2n1i-⅛+⅛)^] 3-fij∙ff2 71, — i⅛ . {nγ- 7⅛)v-

4Ä sιn \*fgh '

Hence the front slope of the tide-wave is steeper at spring than at neap tide, and the compound tide shows itself in the form of an augmentation of the first over-tide ; and the converse statements hold of neap tide. Also mean-water mark is lower and higher alternately up the river at spring tide, and higher and lower alter­nately at neap tide, by a small amount which depends on the dif­ferential tide. With the river which we were considering, the alternation would be so long that it would in actuality be either all lower or all higher.

IV. The Harmonic Analysis.

§ 22. Methods of applying Theory to Practice.

The comparison between tidal observations and tidal theories, and the formation of tables predicting the tidal oscillations of the sea, have been carried out in two different ways, which may be called the “ synthetic ” and the “ analytic. ”

The semi-diurnal rise and fall of tide with the weekly alternation of spring and neap would naturally suggest to the investigator to make his formula conform to the apparent simplicity of the phenomenon. He would seek to represent the height of water by either one or two periodic functions with a variable amplitude ; such a representation is the aim of the synthetic method. That method has been followed by all the great investigators of the past, —Newton, Bernoulli, Maclaurin, Laplace, Lubbock, Whewell, Airy. Since at European ports the two tides which follow one another on any one day are nearly equal, or, in other words, there is scarcely a sensible diurnal tide, these investigators bestowed comparatively little attention to the diurnal tides. If these are neglected, the synthetic method is simple, for a single function suffices to repre­sent the tide. In non-European ports, however, the diurnal tide is sometimes so large as to mask the semi-diurnal, and to make only a single instead of a double high water in twenty-four hours. To represent this diurnal tide in the synthetic method we are compelled to introduce at least one more function. There should also be a third function representing the tides of long period ; but until the last few years these tides have scarcely been considered, and there­fore we shall have little to say of them in explaining the synthetic method. The expression for the tide-generating forces due to either sun or moon consists of three terms, involving the declinations and hour-angles of the planet. One of these terms for each goes through its period approximately twice a day, a second once a day, and the third varies slowly (§ 7). The mathematical basis of the synthetic method consists of a synthesis of the mathematical formulæ. The semi-diurnal term for the moon is fused with that for the sun, and the same process is carried out for the diurnal and slowly varying terms. A mass of tidal observation at a place where the diurnal tide is small, even if, as in all the older observations, it consists merely of heights and times of high and low water, soon shows that the fusion of two simple harmonic or periodic functions is insufficient to represent the state of tide ; and the height and time of high water are found to need corrections for the variations of declination, of motion in right ascension, and of the parallaxes of both bodies.

But when continuous tide-gauges were set up far more extended data than those of the older observations became accessible to the investigator, and more and more corrections were found to be ex­pedient to adapt the formulæ to the facts. A systematic method of utilizing all the data became also a desideratum. This state of matters led Sir W. Thomson to suggest the analytic method.@@1 It is true that the dynamical foundations of that method have always lain below the surface of the synthetic method, and have constantly been appealed to for the theoretical determination of corrections ; nevertheless, we must regard the explicit adoption of the analytic method as a great advance. In this method we conceive the tidal forces or potential due to each disturbing body to be developed into a series of terms each consisting of a constant (determined by the elements of the planet’s orbit and the obliquity of the ecliptic) multiplied by a simple harmonic function of the time. Thus in place of the terms of the synthetic method for the three classes of tides we have an indefinitely long series of terms for each of the three classes. The loss of simplicity in the expression for the forces is far more than counterbalanced by the gain of facility for the discussion of the oscillations of the water. This facility arises from the great dynamical principle of forced oscillations, which we have explained in the historical sketch. Applying this principle, we see that each individual term of the harmonic development of the tide- generating forces corresponds to an oscillation of the sea of the same period, but the amplitude and phase of that oscillation must depend on a network of causes of almost inextricable complication. The analytic method, then, represents the tide at any port by a series of simple harmonic terms whose period is determined from theoretical considerations, but whose amplitude and phase are found from observation. Fortunately the series representing the tidal forces converges with sufficient rapidity to permit us to consider only a moderate number of harmonic terms in the series.

Now it seems likely that the corrections which have been applied in the use of the synthetic method might have been clothed in a more satisfactory and succinct mathematical form had investigators first carried out the harmonic development. In this article we shall therefore invert history and come back on the synthetic method from the analytic, and shall show how the formulæ of correction stated in harmonic language may be made comparable with them in synthetic language. One explanation is expedient before pro­ceeding with the harmonic development. There are certain terms in the tide-generating forces of the moon, depending on the longi­tude of the moon’s nodes, which complete their revolution in 18⋅6 years. Now it has been found practically convenient, in the appli­cation of the harmonic method, to follow the synthetic plan to the extent of classifying together terms whose speed differs only in consequence of the movement of the moon’s node, and at the same time to conceive that there is a small variability in the intensity of the generating forces.

§ 23. Development of Equilibrium Theory of Tides in Terms of the Elements of the Orbits.

Within the limits at our disposal we cannot do more than in­dicate the processes to be followed in this development.

We have already seen in (3) that the expression for the moon’s tide-generating potential is

V=gi>2(cos2z-⅜),

and in (10) that

cos2 z - J = 2frM1M3+2^2 · + 2tfM2M3 + 2∏M1M3

, 3 ^2 + √ι - 2^ M12+M22 - 2M32 +2 3 ‘ 3 ’

where M1,M2,M3 are the direction cosines of the moon referred to axes fixed in the earth. We require to find the functions M1M2, ½(M12 - M22), &c., of the moon’s direction cosines.@@2 Let A, B, C (fig. 2) be the axes fixed in the earth, C being the north pole and AB the equator ; let X, Y, Z be a second set of axes, XY being the plane of the moon’s orbit ; M the projection of the moon in her orbit ; I=ZC, the obliquity of the lunar orbit to the equator ; χ = AX = BCY ; l=MX, the

@@@1 Airy, and after him Chazallon, appear to have been amongst the first to use a kind of harmonic analysis for reducing tidal observations ; but, as Airy did not emancipate himself from the use of hour-angles, declinations, &c., his work can hardly be considered as an example of the analytic method ; see his “ Tides and Waves,” and Hatt’s *Phénomène des Marées,* Paris, 1885.

@@@2 For further details of the analysis of this section, see the Report “On Harmonic Analysis, &c.,” for 1883 to the British Association (Southport).