His problem, in as far as it is now material, is as follows. Let a sphere, of radius a and density w, be made of elastic material whose bulk and rigidity moduli are k and n, and let it be subjected to forces due to a potential wr2S2 per unit volume, where S2 is a surface spherical harmonic of the second order. Then it is required to find the strain of the sphere. We refer the reader to the original sources for the methods of solution applicable to spherical shells and to solid spheres. In order to write Thomson’s solution we put r, λ, l for radius vector, latitude, and longitude, and p, μ, v for the corresponding displacements. Then the solution is as follows :—

(96)∙

For either tidal or rotational stresses S2=τ(sin2λ-⅓), in the case of tides τ=3/2m∕c3, m and c being the moon’s mass and distance, and in the case of rotation τ= -½ω2, ω being the angular velocity about the polar axis. The equation to the surface is found by putting r=a+ρ, where in the expression for ρ we put r=a. Hence from (96) the form of surface is given by r=√1+1τ≡-[1+r⅛⅛]τ'si,15λ^>} -<97>∙

In most solids the bulk modulus is considerably larger than the rigidity modulus, and in this discussion it is sufficient to neglect n compared with k. With this approximation, the ellipticity e of the surface becomes 5wa2

i=W <98>'

Now suppose the sphere to be endued with the power of gravitation, and write „ 19π 2\_g\_

i>wd1' ® 5 a ∖ n

where g is gravity at the surface of the globe. Then, if there were no elasticity, the ellipticity would be given by e=τ∕g, and without gravitation by e=τ∕r. And it may be proved in several ways that, gravity and elasticity co-operating,

e=—=∑.-Lr (100).

r+g g 1+r/g

If n be the rigidity of steel, and if the globe have the size and mean density of the earth, r∕g=2, and with the rigidity of glass r∕g=⅔. Hence the ellipticity of an earth of steel under tide-generating force would be ⅜ of that of a fluid earth, and the similar fraction for glass would be 3/5. If an ocean be superposed on the globe, then, if the globe rises and falls with the tide as though it were fluid, there will obviously be no tide visible to an observer carried up and down with the solid ; and with any degree of rigidity the visible tide will be the excess of the fluid tide above the solid tide. Hence on an earth with rigidity of steel the oceanic tides would be reduced to 2/3, and with rigidity of glass to 2/5 of the tides on a rigid earth.

§ 44. Rigidity of the Earth.

Although the computation of oceanic tides is as yet impossible, it cannot be admitted that perfect rigidity in the earth would aug­ment the tides in the proportion of 5 to 2, although they might perhaps be augmented in the proportion 4 to 3. Thus Thomson concludes that the earth’s mass must have an effective rigidity at least as great as that of steel. If it were true, as was held until recently, that the earth is a fluid ball coated with a crust, that crust must be of fabulous rigidity to resist the tidal surgings of subjacent fluid. Hence we are led to the conclusion that far the larger portion of the earth’s mass, if not all of it, is a solid of great rigidity. Up to the present time the argument by which the tides of long period were proved to have approximately their equilibrium height has generally been accepted without much doubt, but we have (§ 17) shown good cause for rejecting Laplace’s argument, at least for a fortnightly tide. It appeared formerly that, from numeri­cal data as to the heights of the tides of long period, we should be able to compute the actual effective rigidity of the earth’s mass. But from § 18 we see that, although these tides remain incalculable, yet with such oceans as ours the tides of long period must conform much more nearly to the equilibrium laws than do the tides of short period. Thus a comparison of the observed heights of the tides of long period with the equilibrium law still remains of interest, although the evaluation of the earth’s rigidity appears with present data unattainable. Acting on the old belief, Mr G. H. Darwin has compared the lunar fortnightly and monthly tides, as observed for thirty-three years at various Indian and European ports, with the equilibrium theory, and has found that the tide­heights were about two-thirds of the equilibrium height.@@1 From this the conclusion was drawn that the effective rigidity of the earth was as great as that of steel. Whilst, then, this precise com­

parison with the rigidity of steel falls to the ground, the investiga­tion remains as an important confirmation of Thomson’s conclusion as to the great effective rigidity of the earth. When extensive and accurate knowledge of the tides has been attained, the attempted evaluation of the rigidity may conceivably be possible, because there is a minute tide with a period of 18⋅6 years (§ 23, schedule [A, iii.]) of which Laplace’s argument must hold good. Great accu­racy will, however, be necessary, because the height of the tide at the equator only amounts to one-third of an inch, and a preliminary inquiry seems to show that there are other relatively considerable variations of sea-level arising from unexplained causes.@@2

Sir W. Thomson’s solution of the strain of an elastic sphere has been also used to determine what degree of strength the materials of the earth must have in order that the great continental plateaus and mountains may not sink in.@@3 In another investigation it has been shown that local elastic yielding on the coast-lines of conti­nents may produce an augmentation of apparent tide in certain places on account of the flexure of the upper strata, when a great weight of water is added and subtracted from the adjacent oceanic area at high and low tide.@@4 There is reason to believe that such flexure has actually been observed by a delicate form of level on the coast of the Bay of Biscay.@@5

§ 45. Viscous and Elastico- Viscous Tides.

It might be supposed that the earth is composed of a viscous fluid of great stiffness, or that it possesses an elasticity which breaks down under continued stress. Both these hypotheses have been considered, and the results are confirmatory of the conclusion that the earth is made of very stiff material.@@6 These problems appear to have been worthy of attack, although the existence of measurable oceanic tides of long period negatives the adoption of the hypothesis of true viscosity, at least under stresses comparable with tide-generating forces.

If a sphere of radius a, density w, viscosity modulus v, be under the action of forces due to a potential per unit volume wr2S2 cos nt, so that n is the speed of the tide, the solution of the problem shows that the tide of the sphere is expressed by

c~ cos e cos lnt - e) (101),

1 . t 19υ g

where Une=.-, t=j-i, g=J s-.

Thus the tides of the viscous globe are to the equilibrium tides of a fluid globe as cos ϵ to unity, and there is a retardation ϵ∣n of the time of high tide after the passage of the tide-generator over the meridian. Further, by arguments similar to that applied in the case of elastic tides, it is found that oceanic tides are reduced by the yielding in the proportion of sin ϵ to unity, and that there is an apparent acceleration of the time of high water by (⅛7r-e)∕n. It appears by numerical calculation that, in order that the oceanic semi-diurnal tide may have a value equal to two-thirds of the full amount on a rigid globe, the stiffness of the globe must be about twenty thousand times as great as that of pitch at freezing temperature, when it is hard and brittle. We must here pass by the results of the hypothesis of an elasticity degrading under the influence of continued stress.

IX. Tidal Friction.

§ 46. General Explanation.

The investigation of the tides of a viscous sphere has led us to the consideration of a frictionally retarded tide. The effects of tidal friction are of such general interest that we give a sketch of the principal results without the aid of mathematical symbols. In fig. 8 the paper is supposed to be the plane of the orbit of a satellite M revolving in the direction of the arrow about the planet C, which rotates in the direction of the arrow about an axis perpendicular to the paper. The rotation of the planet is supposed to be more rapid than that of the satellite, so that the day is shorter than the month. Let us suppose that the planet is either entirely fluid, or has an ocean of such depth that it is high water under or nearly under the satellite. When there is no friction, with the satellite at m, the planet is elongated into the ellipsoidal shape shown, cutting the mean sphere, which is dotted. But, when there is friction in the fluid motion, the tide is retarded, and high tide occurs after the satellite has passed the meridian. Then, if we keep the same figure to represent the tidal elongation, the satellite must be at M, instead of at m. If we number the four quadrants as shown, the satellite must be in quadrant 1. The protuberance P is nearer to the satellite than P', and the deficiency Q is further away than the deficiency

@@@1 Thomson and Tait, *Nat. Phil.,* vol. i. pt. ii., 1883, § 847 *sq.*

@@@2 Darwin, “On 19-yearly Tide at Karachi,” in *Brit. Assoc. Report,* 1886.

@@@3 G. H. Darwin, *Phil. Trans.,* pt. i., 1882, p. 187, with correction, *Proc. Roy. Soc.,* 1885.

@@@4 Id., *Brit. Assoc. Rep.,* 1882, *or Phil. Mag.,* 1882.

@@@5 D’Abbadie, *Annales Soc. Sc. de Bruxelles,* 1881, or quotation by Darwin, *loc. cit.*

@@@6 G. H. Darwin, *Phil. Trans.,* pt. i., 1879, p. 1 ; see also Lamb, “ On the Oscillations of a viscous Spheroid,” *Proc. Land. Math. Soc.,* Nov. 1881, p. 51.