Q'. Hence the resultant action of the planet on the satellite must be in some such direction as MN. The action of the satellite on the planet is equal and opposite, and the force in NM, not being through the planet’s centre, must produce a retarding couple on the planet’s rotation, the magnitude of which depends on the length of the arm CN. This tidal frictional couple varies as the height of the tide, and also depends on the sa­tellite’s distance ; its in­tensity in fact varies as the square of the tide- generating force, and therefore as the inverse sixth power of the satellite’s distance. Thus tidal friction must retard the planetary rotation. Let us now con­sider its effect on the satellite. If the force acting on M be resolved along and perpendicular to the direction CM, the perpendicular component tends to accelerate the satellite’s velocity. It alone would carry the satellite further from C than it would be dragged back by the central force towards C. The satellite would describe a spiral, the coils of which would be very nearly circular and very nearly coincident. If now we resolve the central component force along CM tangentially and perpendicular to the spiral, the tangential component tends to retard the velocity of the satellite, whereas the disturbing force, already considered, tends to accelerate it. With the gravitational law of force between the two bodies the retarda­tion must prevail over the acceleration.@@1 The moment of mo­mentum of the whole system remains unchanged, and that of the planetary rotation diminishes, so that the orbital moment of momentum must increase ; now orbital moment of momentum in­creases with increasing distance and diminishing linear and angular velocity of the satellite. The action of tidal friction may appear somewhat paradoxical, but it is the exact converse of the accelera­tion of the linear and angular velocity and the diminution of dis­tance of a satellite moving through a resisting medium. The latter result is generally more familiar than the action of tidal friction, and it may help the reader to realize the result in the present case. Tidal friction then diminishes planetary rotation, increases the satellite’s distance, and diminishes the orbital angular velocity. The comparative rate of diminution of the two angular velocities is generally very different. If the satellite be close to the planet the rate of increase of the satellite’s periodic time or month is large compared with the rate of increase of the period of planetary rota­tion or day ; but if the satellite is far off the converse is true. Hence, if the satellite starts very near the planet, with the month a little longer than the day, as the satellite recedes the month soon increases, so that it contains many days. The number of days in the month attains a maximum and then diminishes. Finally the two angular velocities subside to a second identity, the day and month being identical and both very long.

We have supposed that the ocean is of such depth that the tides are direct ; if, however, they are inverted, with low water under or nearly under the satellite, friction, instead of retarding, accelerates the tide ; and it would be easy by drawing another figure to see that the whole of the above conclusions hold equally true with inverted tides.

§ 47. Exact Investigation of the Secular Effects of Tidal Friction.

The general conclusions of the last section are of such wide in­terest that we proceed to a rigorous discussion of the principal effects of tidal friction in the elementary case of the circular orbit. In order, however, to abridge the investigation we shall only consider the case when the planetary rotation is more rapid than the satel­lite’s orbital motion.

Suppose an attractive particle or satellite of mass m to be moving in a circular orbit, with an angular velocity Ω, round a planet of mass M, and suppose the planet to be rotating about an axis perpendicular to the plane of the orbit, with an angular velocity n ; suppose, also, the mass of the planet to be partially or wholly imperfectly elastic or viscous, or that there are oceans on the sur­face of the planet ; then the attraction of the satellite must produce a relative motion in the parts of the planet, and that motion must be subject to friction, or, in other words, there must be frictional tides of some sort or other. The system must accordingly be losing energy by friction, and its configuration must change in such a way that its whole energy diminishes. Such a system does not differ much from those of actual planets and satellites, and, therefore, the results deduced in this hypothetical case must agree pretty closely with the actual course of evolution, provided that time enough has

been and will be given for such changes. Let C be the moment of inertia of the planet about its axis of rotation, r the distance of the satellite from the centre of the planet, h the resultant moment of momentum of the whole system, e the whole energy, both kinetic and potential, of the system. It is assumed that the figure of the planet and the distribution of its internal density are such that the attraction of the satellite causes no couple about any axis perpen­dicular to that of rotation. A special system of units of mass, length, and time will now be adopted such that the analytical re­sults are reduced to their simplest forms. Let the unit of mass be Mm∣(M+m). Let the unit of length γ be such a distance that the moment of inertia of the planet about its axis of rotation may be equal to the moment of inertia of the planet and satellite, treated as particles, about their centre of inertia, when distant γ apart from one another. This condition gives

4⅛),÷∙(⅛)⅛ whence . Η2⅛ιi}4∙

Let the unit of time τ be the time in which the satellite revolves through 57o⋅3 about the planet, when the satellite’s radius vector is equal to γ. In this case 1∕τ is the satellite’s orbital angular velocity, and by the law of periodic times we have τ-2γ3=μ(M+m),

where μ is the attraction between unit masses at unit distance. Then by substitution for γ

\_ f c3(JΓ+m)∙⅜ ⅜

τ I μ∖Mmf ∫

This system of units will be found to make the three following functions each equal to unity, viz., μ½Mm (M+m)-½, and C.

The units are in fact derived from the consideration that these functions are each to be unity. In the case of the earth and moon, if we take the moon’s mass as 1/82d of the earth’s and the earth’s moment of inertia as ⅓Ma2 (as is very nearly the case), it may easily be shown that the unit of mass is 1/83 of the earth’s mass, the unit of length 5⋅26 earth’s radii or 33,506 kilomètres (20,807 miles), and the unit of time 2 hrs. 41 minutes. In these units the present angular velocity of the earth’s diurnal rotation is expressed by ⋅7044, and the moon’s present radius vector by 11⋅454. The two bodies being supposed to revolve in circles about their common centre of inertia with an angular velocity Ω, the moment of momen­tum of orbital motion is

w ∞L-y0+j√

∖M+mJ ∖M+mj M+m

Then, by the law of periodic times in a circular orbit, Ω2r3=μ(M+m) ;

whence Ωr2 =μ½(M+m)½r½.

The moment of momentum of orbital motion

= μ½Mm( M + m)-½r½,

and in the special units this is equal to r½. The moment of momentum of the planet’s rotation is Cn, and (C=1 in the special units. Therefore h=n+r½ (102).

Since the moon’s present radius vector is 11⋅454, it follows that the orbital momentum of the moon is 3⋅384. Adding to this the rotational momentum of the earth, which is ⋅704, we obtain 4⋅088 for the total moment of momentum of the moon and earth. The ratio of the orbital to the rotational momentum is 4∙80, so that the total moment of momentum of the system would, but for the obliquity of the ecliptic, be 5⋅80 times that of the earth’s rotation. Again, the kinetic energy of orbital motion is

The kinetic energy of the planet’s rotation is ½Cn2. The potential energy of the system is - μMm∣r. Adding the three energies to­gether, and transforming into the special units, we have 2e=n2-1∕r (103).

Now let x=r½, y=n, Y=2e.

It will be noticed that x, the moment of momentum of orbital motion, is equal to the square root of the satellite’s distance from the planet. Then equations (102) and (103) become

h=y+x (104).

Y=y2- 1∕z2=(h - x)2 - 1∕x2 (105).

(104) is the equation of conservation of moment of momentum, or, shortly, the equation of momentum ; (105) is the equation of energy.

Now consider a system started with given positive moment of momentum h ; and we have all sorts of ways in which it may be started. If the two rotations be of opposite kinds, it is clear that we may start the system with any amount of energy however great, but the true maxima and minima of energy compatible with the given moment of momentum are supplied by dY/dx=0, or x-h + 1∕x3=0,

that is to say, x4-hx3 + 1 = 0 (106).

We shall presently see that this quartic has either two real roots

@@@1 This way of presenting the action of tidal friction is due to Professor Stokes.