by the use of the former. He was the first to calculate sin*ϕ* from the equation sin*ϕ*∕cos*ϕ* = *k,* and he also made a table of the lengths of shadows of a vertical object of height 12 for altitudes 1°, 2°, . .. of the sun; this is a sort of cotangent table. He was acquainted, not only with the triangle formulae in the *Almagest,* but also with the formula cos *a* = cos *b* cos *c* + sin *b* sin *c* cos *A* for a spherical triangle *ABC.* Abú 'l-Wafá of Baghdad (b. 940) was the first to introduce the tangent as an independent function : his “ umbra ” is the half of the tangent of the double arc, and the secant he defines as the “diameter umbræ.” He employed the umbra to find the angle from a table and not merely as an abbreviation for sin/cos ; this improvement was, however, afterwards forgotten, and the tangent was re-invented in the 15th century. Ibn Yúnos of Cairo, who died in 1008, showed even more skill than Al-Battání in the solution of problems in spherical trigonometry and gave improved approximate formulæ for the calculation of sines. Among the West Arabs, Abú Mohammed Jābir b. Aflah, known as Geber b. Aflah, who lived at Seville in the 11th century, wrote an astronomy in nine books, which was translated into Latin in the 12th century by Gerard of Cremona and was published in 1534. The first book con­tains a trigonometry which is a considerable improvement on that in the *Almagest.* He gave proofs of the formulæ for right-angled spherical triangles, depending on a rule of four quantities, instead of Ptolemy’s rule of six quantities. The formulæ cos *B = cos b* sin *A,* cos *c* = cot *A* cot *B,* in a triangle of which *C* is a right angle had escaped the notice of Ptolemy and were given for the first time by Geber. Strangely enough, he made no progress in plane trigono­metry. Arrachel, a Spanish Arab who lived in the 12th century, wrote a work of which we have an analysis by Purbach, in which, like the Indians, he made the sine and the arc for the value 3° 45' coincide.

Purbach (1423-1461), professor of mathematics at Vienna, wrote a work entitled *Tractatus super propositiones Ptole- mæi de sinubus et chordis* (Nuremberg, 1541). This treatise consists of a development of Arrachel’s method of inter­polation for the calculation of tables of sines, and was pub­lished by Regiomontanus at the end of one of his works. Johannes Müller (1436-1476), known as Regiomontanus (*q.v.*), was a pupil of Purbach and taught astronomy at Padua ; he wrote an exposition of the *Almagest* and a more important work, *De triangulis planis et sphericis cum tabulis sinuum,* which was published in 1533, a later edition ap­pearing in 1561. He re-invented the tangent and calcu­lated a table of tangents for each degree, but did not make any practical applications of this table, and did not use formulæ involving the tangent. His work was the first complete European treatise on trigonometry, and contains a number of interesting problems ; but his methods were in some respects behind those of the Arabs. Copernicus (1473-1543) gave the first simple demonstration of the fundamental formula of spherical trigonometry ; the *Trigo- nometria Copernici* was published by Rheticus in 1542. George Joachim (1514-1576), known as Rheticus *(q.v.)* wrote *Opus Palatinum de triangulis* (see Tables, p. 9 above), which contains tables of sines, tangents, and secants of arcs at intervals of 10" from 0° to 90°. His method of calculation depends upon the formulæ which give sin *nα* and cos *nα* in terms of the sines and cosines of (*n* - 1 )α and (*n* - 2)α ; thus these formulæ may be regarded as due to him. Rheticus found the formulæ for the sines of the half and third of an angle in terms of the sine of the whole angle. In 1599 there appeared an important work by Pitiscus (1561-1613), entitled *Trigonometries seu de dimen­sione triangulorum* ; this contained several important theo­rems on the trigonometrical functions of two angles, some of which had been given before by Finck, Landsberg, and Adriaan van Roomen. François Viète or Vieta (*q.v.*) (1540-1603) employed the equation (2 cos½*φ*)3 - 3(2cos½φ) = 2 cos *φ* to solve the cubic *x*3 *— 3a2x* = *a*2*b*(*a* > ½*b)* ; he ob­tained, however, only one root of the cubic. In 1593 Van Roomen proposed, as a problem for all mathematicians, to solve the equation

45*y* — 3795*y*3 + 95634*y*5 —... + 945*y*41 — 45*y*43 + *y45 = C.*

Viète gave *y* = 2 sin 1/45*φ*, where (7 = 2 sin *φ*, as a solution, and also twenty-two of the other solutions, but he failed to obtain the negative roots. In his work *Ad angulares sectiones* Viète gave formulæ for the chords of multiples of a given arc in terms of the chord of the simple arc.

A new stage in the development of the science was commenced after Napier’s invention of logarithms in 1614. Napier also simplified the solution of spherical triangles by his well-known analogies and by his rules for the solution of right-angled triangles. The first tables of logarithmic sines and tangents were constructed by Edmund Gunter (1581-1626), professor of astronomy at Gresham College, London ; he was also the first to employ the expressions cosine, cotangent, and cosecant for the sine, tangent, and secant of the complement of an arc. A treatise by Albert Girard (1590-1634), published at The Hague in 1626, con­tains the theorems which give areas of spherical triangles and polygons, and applications of the properties of the supplementary triangles to the reduction of the number of different cases in the solution of spherical triangles. He used the notation sin, tan, sec for the sine, tangent, and secant of an arc. In the second half of the 17th century the theory of infinite series was developed by Wallis, Gregory, Mercator, and afterwards by Newton and Leibnitz. In the *Analysis per aequationes numero terminorum infinitas,* which was written before 1669, Newton gave the series for the arc in powers of its sine ; from this he obtained the series for the sine and cosine in powers of the arc ; but these series were given in such a form that the law of the formation of the coefficients was hidden. James Gregory discovered in 1670 the series for the arc in powers of the tangent and for the tangent and secant in powers of the arc. The first of these series was also discovered inde­pendently by Leibnitz in 1673, and published without proof in the *Acta eruditorum* for 1682. The series for the sine in powers of the arc he published in 1693; this he obtained by differentiation of a series with undetermined coefficients.

In the 18th century the science began to take a more analytical form ; evidence of this is given in the works of Kresa in 1720 and Mayer in 1727. Oppel’s *Analysis triangulorum* (1746) was the first complete work on ana­lytical trigonometry. None of these mathematicians used the notation sin, cos, tan, which is the more surprising in the case of Oppel, since Euler had in 1744 employed it in a memoir in the *Acta eruditorum.* John Bernoulli was the first to obtain real results by the use of the symbol

√-1 ; he published in 1712 the general formula for tan *nφ* in terms of tanφ, which he obtained by means of trans­formation of the arc into imaginary logarithms. The greatest advance was, however, made by Euler, who brought the science in all essential respects into the state in which it is at present. He introduced the present nota­tion into general use, whereas until his time the trigono­metrical functions had been, except by Girard, indicated by special letters, and had been regarded as certain straight lines the absolute lengths of which depended on the radius of the circle in which they were drawn. Euler’s great im­provement consisted in his regarding the sine, cosine, &c., as functions of the angle only, thereby giving to equations connecting these functions a purely analytical interpreta­tion, instead of a geometrical one as heretofore. The