exponential values of the sine and cosine, De Moivre’s theorem, and a great number of other analytical properties of the trigonometrical functions are due to Euler, most of whose writings are to be found in the *Memoirs* of the St Petersburg Academy.

The preceding sketch has been mainly drawn from the following sources: — Cantor, *Gesch. d. Math.;* Hankel, *Gesch. d. Math.·,* Marie, *Hist. des sc. math.* ; Suter, *Gesch. d. Math.* ; Klügel, *Math. Wörterbuch.*

*Plane Trigonometry.*

Imagine a straight line terminated at a fixed point *O,* and initially coincident with a fixed straight line *OA,* to revolve round *O,* and finally to take up any position *OB.* We shall suppose that, when this re­volving straight line is turning in one direction, say that opposite to that in which the hands of a clock turn, it is describing a positive angle, and when it is turning in the other direction it is describing a negative angle. Before finally taking up the position *OB* the straight line may have passed any num­ber of times through the position *OB,* making any number of complete revo­lutions round *O* in either direction. Each time that the straight line makes a complete revolution round *O* we consider it to have described four right angles, taken with the positive or negative sign according to the direction in which it has revolved ; thus, when it stops in the position *OB,* it may have revolved through any one of an infinite number of positive or negative angles any two of which differ from one another by a positive or negative multiple of four right angles, and all of which have the same bounding lines *OA* and *OB.* If *OB'* is the final position of the revolving line, the smallest positive angle which can have been described is that described by the revolv­ing line making more than one-half and less than the whole of a complete revolution, so that in this case we have a positive angle greater than two and less than four right angles. We have thus shown how we may conceive an angle not restricted to less than two right angles, but of any positive or negative magnitude, to be generated.

Two systems of numerical measurement of angular magnitudes are in ordinary use. For practical measurements the sexagesimal system is the one employed : the ninetieth part of a right angle is taken as the unit and is called a degree ; the degree is divided into sixty equal parts called minutes ; and the minute into sixty equal parts called seconds ; angles smaller than a second are usually measured as decimals of a second, the “thirds,” “fourths,” &c., not being in ordinary use. In the common notation an angle, for ex­ample, of 120 degrees, 17 minutes, and 14∙36 seconds is written 120o 17' 14"∙36. The decimal system measurement of angles has never come into ordinary use. In analytical trigonometry the circular measure of an angle is employed. In this system the unit angle is the angle subtended at the centre of a circle by an arc equal in length to the radius. The constancy of this angle follows from the geometrical propositions—(1) the circumferences of different circles vary as their radii ; (2) in the same circle angles at the centre are proportional to the arcs which subtend them. It thus follows that the unit mentioned above is an angle independent of the particular circle used in defining it. The constant ratio of the circumference of a circle to its diameter is a quantity incommensurable with unity, usually denoted by π. We shall indicate later on (p. 571 *sq.* ) some of the methods which have been employed to approximate to the value of this quantity. Its value to 20 places is 3 ∙14159265358979323846 ; its reciprocal to the same number of places is ∙31830988618379067153. In circular measure every angle is measured by the ratio which it bears to the unit angle. Two light angles are measured by the quantity *π,* and, since the same angle is 180o, we see that the number of degrees in an angle of circular measure *θ* is obtained from the formula 180 × *θ∕π.* The value of the unit of circular measure has been found to 41 places of decimals by Glaisher (*Proc. London Math. Sοc.,* vol. iv.) ; the value of 1/π from which the unit can be easily calculated, is given to 140 places of decimals in *Grunert's Archiv,* vol. i., 1841. To 10 decimal places the value of the unit angle is 57° 17'44"∙8062470964. The unit of circular measure is too large to be convenient for practical purposes, but its use introduces a simplification into the series in analytical trigonometry, owing to the fact that the sine of an angle and the angle itself in this measure, when the magnitude of the angle is indefinitely diminished, are ultimately in a ratio of equality.

If a point moves from a position *A* to another position *B* on a straight line, it has described a length *AB* of the straight line. It is convenient to have a simple mode of indicating in which direction on the straight line the length *AB* has been described ; this may be done by supposing that a point moving in one specified direction is describing a positive length, and when moving in the opposite direction a negative length. Thus, if a point moving from *A* to *B ∣* is moving in the positive direction, we consider the length *AB* as : positive ; and, since a point moving from *B* to *A* is moving in the negative direction, we consider the length *BA* as negative. Hence any portion of an infinite straight line is considered to be positive or negative according to the direction in which we suppose this portion to be described by a moving point ; which direction is the positive one is, of course, a matter of convention.

If perpendiculars *AL, BM* be drawn from two points *A, B* on any straight line, not necessarily in the same plane with *AB,* the length *LM,* taken with the positive or negative sign according to the convention as stated above, is called the projection of *AB* on the given straight line ; the projection of *BA* being *ML* has the opposite sign to the projection of *AB.* If two points *A, B* be joined by a number of lines in any manner, the algebraical sum of the projections of all these lines is *LM,—*that is, the same as the pro­jection of *AB.* Hence the sum of the projections of all the sides of any closed polygon, not necessarily plane, on any straight line, is zero. This principle of projections we shall apply below to ob­tain some of the most important propositions in trigonometry.

Let us now return to the conception of the generation of an angle as in fig. 1. Draw *BOB'* at right angles to and equal to *AA'.*

We shall suppose that the direction from *A'* to *A* is the’ positive one for the straight line *AO A',* and that from *B'* to *B* for *BOB'.* Suppose *OP* of fixed length, equal to *OA,* and let *PM, PN* be drawn perpendicular to *A'A, B'B* respectively ; then *OM* and *ON,* taken with their proper signs, are the projec­tions of *OP* on *A'A* and *B'B.* The ratio of the projection of *OP* on *B'B* to the absolute length of *OP* is dependent only on the magni­tude of the angle *POA,* and is called the sine of that angle ; the ratio of the projection of *OP* on *A'A* to the length *OP* is called the cosine of the angle *POA.* The ratio of the sine of an angle to its cosine is called the tangent of the angle, and that of the cosine to the sine the cotangent of the angle ; the reciprocal of the cosine is called the secant, and that of the sine the cosecant of the angle. These functions of an angle of magnitude *a* are denoted by sin *a,* cos *a,* tan *a,* cot *a,* sec *a*, cosec *a* respectively. If any straight line *RS* be drawn parallel to *OP,* the projection of *RS* on either of the straight lines *A'A, B'B* can be easily seen to bear to *RS* the same ratios which the corresponding projections of *OP* bear to *OP* : thus, if *a* be the angle which *RS* makes with *A'A,* the projections of *RS* on *A'A, B'B* are *RS* cos *a* and *RS* sin *a* respectively, where *RS* denotes the absolute length *RS.* It must be observed that the line *SR* is to be considered as parallel not to *OP* but to *OP",* and therefore makes an angle π *+ a* with *A'A ;* this is consistent with the fact that the projections of *SR* are of opposite sign to those of *RS.* By observing the signs of the projections of *OP* for the positions *P, P', P", P"'* of *P* we see that the sine and cosine of the angle *POA* are both positive ; the sine of the angle *P'OA* is posi­tive and its cosine is negative ; both the sine and the cosine of the angle *P''OA* are negative ; and the sine of the angle *P"OA* is negative and its cosine positive. If α be the numerical value of the smallest angle of which *OP* and *OA* are boundaries, we see that, since these straight lines also bound all the angles *2nπ + a,* where *n* is any positive or negative integer, the sines and cosines of all these angles are the same as the sine and cosine of *a.* Hence the sine of any angle 2nπ + *a* is positive if *a* is between 0 and π and negative if *a* is between π and 2π, and the cosine of the same angle is positive if α is between 0 and ½π or 3/2 *π* and 2π and negative if α is between ½π and 3/2π.

In fig. 2 if the angle *POA* is α, the angle *P"''\OA* is - α, *P'OA* is π - *a*, *P''OA* is π + *a*, *POB* is π/2 - *a*. By observing the signs of the projections we see that

sin ( - *a*)= - sin a, sin (π- *a*) = sin *a,* sin(π + *a*)=-sin*a*, cos ( - *a*) = cos *a*, cos (π - *a*) = - cos *a,* cos (π + *a*) = - cos *a*, sin(½π-*a*) = cos*a*, cos(½π - *a*) = sin a.

Also sin(½π +*a*) = sin(π - ½π - *a)=* sin(½π-*a*)= cos *a*, cos(½π + *a*) = cos(π - ½π - *a*)= -cos(½π-*a*)= - sin *a.*

From these equations we have tan(*-a)= -* tan *α*, tan(π - α)= — tan *a,* tan (π + *α*) = — tan *α*, tan (½π — *a*) = cot a, tan(½π + *a) =* — cot a, with corresponding equations for the cotangent.

The only angles for which the projection of OP on B'B is the same as for the given angle POA ( = α) are the two sets of angles bounded by OP, OA and OP', OA ; these angles are 2nπ + α and 2*n*π + π - *α*, and are all included in the formula *r*π + ( - 1)rα, where r is any integer ; this therefore is the formula for all angles having the same sine as *α*. The only angles which have the same cosine as *α* are those bounded by OA, OP and OA, OP"', and these are all included in the formula 2*n*π±*α*. Similarly it can be shown