that iiιr + α includes all the angles which have the same tangent as a.

From the Pythagorean theorem, the sum of the squares of the projections of any straight line upon two straight lines at right angles to one another is equal to the square on the projected line, we get sin2α + cos2α = l, and from this by the help of the definitions of the other functions we deduce the relations 1+ tan2α=sec2α, 1+ cot2a = cosec2a. We have now six relations between the six functions ; these enable us to express any five of these functions in terms of the sixth. The following table shows the values of the trigonometrical functions of the angles 0, ⅜τr, τr, ⅜ττ, *2ττ,* and the signs of the functions of angles between these values ; *I* denotes numerical increase and *D* numerical decrease.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Angle .... | 0 | 0.. . Jπ |  | ⅛7Γ... 7Γ | *π* | 7Γ. . . ^7Γ | $7T | ⅞7Γ . . . 27Γ | *2π* |
| Sine | 0 | ÷∕ | 1 | *+D* | 0 | *-I* | - 1 | *-D* | 0 |
| Cosine .... | 1 | *+D* | 0 | *-I* | - 1 | *-D* | 0 | *+ι* | 1 |
| Tangent .. | 0 | *+1* | ±∞ | *-D* | 0 | *+ι* | ±(fc | *-D* | 0 |
| Cotangent | i ∞ | *+D* | 0 | *-I* | ±∞ | *+D* | 0 | *-I* | ±∞ |
| Secant .... | 1 | *+1* | ±00 | *-D* | - 1 | *-I* | ± X | *+D* | 1 |
| Cosecant .. | i ∞ | *+D* | 1 | *+1* | ±∞ | *-D* | - 1 | *-I* | ±00 |

The correctness of the table may be verified from the figure by con­sidering the magnitudes of the projections of *OP* for different positions.

The following table shows the sine and cosine of some angles for which the values of the functions may be obtained geometrically :—

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| 7Γ  12 | 15∙ | sine √6-√2 4 | cosine ∖76±λ∕2  4 | 75’ | ⅛7r |
| 7Γ  10 | 18’ | √5^1  4 | √10+2√5  4 | 72’ | 5τ |
| *π*  6 | 30’ | 1  2 | 2 | 60’ |  |
| 7Γ  5 | 36’ | √10-2√5  4 | √5+l  4 | 54 | ≡π∙  10 |
| 7Γ  4 | 45’ | 1 ∙√2^ cosine | 1  √2 sine | 45’ | 1  4ir |

These are obtained as follows. (1) The sine and cosine of this angle are equal to one another, since sin j = cos (^ - ~) ; and since the sum of the squares of the sine and cosine is unity each is ~=∙ (2) and ≈∙ Consider an equilateral triangle ; the projection of one side on another is obviously half a side ; hence the cosine of an angle of the triangle is or cos J=and from this the sine is found. (3) In the triangle constructed in Eue. iv.

1U 0 O 1U 27γ 7γ

10 each angle at the base is =-, and the vertical angle is If *a* be a side and *b* the base, we have by the construction α(α *~b) = b2 ;* hence 2⅛=a(√5 -1) ; the sine of *f-* is or \* and cos^^ is 66 *∖J ⅜* I 1 *ΊΓ κ)ΙΓ* iθ *2∙(c* 4 O

2£=—j—· (4) jg, jg. Consider a right-angled triangle, hav­

ing an angle ⅜7τ. Bisect this angle, then the opposite side is cut by the bisector in the ratio of √3 to 2 ; hence the length of the smaller segment is to that of the whole in the ratio of √3to √3 + 2, therefore tan ⅛r = —tan ⅜7r or tan 1⅛τr= 2 - √3, and from this *∖j* 3 + 2

we can obtain sin ⅛7r and cos

r.

Draw a straight line *OD* making any angle *A* with a fixed straight line *OA,* and draw *OF* making an angle *B* with *OD,* this angle being measured positively in the same direction as *A* ; draw *FE* a perpendicular on *DO* (produced if necessary). The projection of *OF* on *OA* is the sum of the projections of *OE* and *EF* on *OA.* Now *OE* is the projection of *OF* on *DO,* and is therefore equal to *OF* cos *B,* and *EF* is the projection of *OF* on a straight line making an angle + ⅛7τ with *OD,* and is therefore equal to *OFsvιιB* ; hence

*OF cos(A + B)= OE* cos *A + EF* cos (⅛7r + *A )*

*= OF* (cos *A* cos *B -* sin *A* sin *By),*

or cos *(A + B~)=cosA* cos *B-* sin∠4 sin 2?.

The angles *A, B* are absolutely unrestricted in magnitude, and thus this formula is perfectly general. We may change the sign of *B,* thus *cos (A — B) = cosA* cos ( — *B) -* sin *A* sin ( - *B), or cos (A -B) = cos A cos B*+sin *A* sin *B.*

If we projected the sides of the triangle *OEF* on a straight line making an angle +⅜7τ with *OA* we should obtain the formulæ

sin (^4±5) = sin *A* cos Z>÷cos *A* sin *B,* which are really contained in the cosine formula, since we may put ⅛7r - *B* for *B.* The formulæ

z.,τ,x tan^4±tanN *cot A* cot-Cψl

⅛μ⅛~~lτtelj~~~~,~~~~t~~~~,y~~ cot(∠⅛j)≈ cot^faot÷.

are immediately deducible from the above formulæ. The equations

sin C,+sin *D=2* sin⅜ *(tC+D)* cos∣ *{C-Dy),*

sin *C -* sin *D=2* sin⅛ (C, - D) cos∣ (C,+ Z>),

cos Z> + cos *C=2* cos⅜ (C,+Z>) cos⅜ *(C-D),*

cos *D -* cos *C=2* sin⅛ (C,+Z>) sin⅜ *∖C- D~),*

may be obtained directly by the method of projections. Take two equal straight lines *OC, OD,* making angles *C, D* with *OA,* and draw *OE* perpendicular to *CD.* The angle which *OE* makes with *OA* is ⅛(C,+Z>) and that which *DC* makes is *⅛(π + C+D)∙,* the angle *COE* is *⅛(C-D).* The sum of the projections of *OD* and *DE* on *OA* is equal to that of *OE,* and the sum of the projections of *OD* and *DE* is equal to that of *OC* ; hence the sum of the projections of *OC* and *OD* is twice that of *OE,* or cos *C* + cos *D=2* cos ⅜(C'+Z>) cos⅛(t7-Z)). The difference of the projections of *OD* and *OC* on *OA* is equal to that of *ED,* hence we have the formula cos *D -* cos *C=2* sin ⅜(C'+D) sin ⅜(C,- *Dy).* The other two formulæ will be obtained by projecting on a straight line inclined at an angle + ⅛7r to *OA.*

As another example of the use of projections, we will find the sum of the series cosα + cos(α + ∕3)+ cos(α + 2∕3)+...+cos(α + 7i-1/3). Suppose an unclosed polygon each angle of which is *π - β* to be in­scribed in a circle, and let *A1, A2, A3, ..., An* be 7i + l consecutive angular points ; let *D* be the diameter of the circle ; and suppose a straight line drawn making an angle *a* with *AA1,* then *a + β, a + 2β,...* are the angles it makes with *A1A2, A2A3,...;* we have by projections

*AAn* cos^α + —= ^4√41(cos *a* + cos *a + β* + ..'. + cos a + *n* - 1 *β∖* also Λ∠tι=Z>sin^, *AAn=Dsin~ ;*

2 2 hence the sum of the series of cosines is cos ( a+⅛ —sin cosec By a double application oι the addition formulæ we may obtain the formulæ

sin *{A1 + A2 + A3) =* sin *A1* cos *A2* cos *A3* + cos *A1* sin *A2* cos *A3*

+ cos *A1* cos *A2* sin *A3 -* sin *A1* sin *A2* sin *A3 ;*

cos (J1 + J2+J3)=cos √∕∣ cos *A2* cos *A3* - cos *A1* sin *A2* sin *A3*

- sin *A1* cos *A2* sin *A3 -* sin *A1* sin *A2* cos *A3.*

We can by induction extend these formulæ to the case of *n* angles. Assume sin (Λ1 + Λ2+ ... *+ An) = S1- S3 + S5 - ...*

cos *(A, + A*2 + ... + *An)* = <So — *S2 + Si — ...*

where *Sr* denotes the sum of the products of the sines of *r* of the angles and the cosines of the remaining *n-r* angles ; then we have sin (Λ1 + Λ2+ ... *+ An + An+1)=cos An+1{S1- S3 + S&- ...)*

+ sin Λ+ι('so - -⅞ + *S4 - ... ).* The right-hand side of this equation may be written

(<S,1 cos Jn+1 + S,o sin Λn+1) - (⅞ cos ∠fn+1 + *S2* sin *AnH) +...,* or 0,'1-0,'3+...

where *S,r* denotes the quantity which corresponds for *n* +1 angles to *Sr* for *n* angles ; similarly we may proceed with the cosine for­mula. The theorems are true for n = 2 and n=3 ; thus they are true generally. The formulæ

cos *2A = cos2A -* sin24 = 2 cos2√4 -1 = 1-2 sin2∠4,

sin *2A —* 2 sin *A* cos *A,* tan *2A = ~~f ^~~~~an~~ ~~f~~*

1 - tan2A sin *3 A* =3 sin√4 - 4 sin3√4, cos 3∠4 = 4cosM - 3cos^4,

sin *nA =n cosn~'iA* sin *A - 7-—eosn~3A sinsA + ...*

+ ( - cιιt~2^ cos-2-L4 sin2\*∙+¼

cos *nA* = cos *nA - cosn~2A* sin2J + ...

~~+ (~~ ~~\_~~ ~~ιr~~~~n(T-l)...0~~~~t~~~~-~~~~2~~~~7∙~~ ~~+ 1~~~~)~~ ~~cosn~~~~,~~~~2~~~~^~~ ~~sin2~~~~^~~ ~~+~~ ~~..~~

may all be deduced from the addition formulæ by making the angles all equal. From the last two formulæ we obtain by division tan *nA*

~~.to ^-⅛-g~~~~,~~~~!-~~~~2~~~~>~~toU+... toMU+...

l-⅛⅛i>toU÷...-κ-l>^,-1>∙^-2⅛ι>to^+...

In the particular case of n=3 we have tan 3√4 = ~~A tan~~~~2~~~~√4~~