The values of sin ⅛*A,* cos½A, tan⅜Λ are given in terms of cos J by the formulæ

. , , , „,„/1-cos√4∖1 1 , . 1.9∕l+cosJ∖i

sin⅛Λ = (-l)p(^ g J , cos⅛Λ = (-l),^ ) ’

**, . , ιxr∕l-COsJ∖⅛**

tan⅛Λ = (-l) (rj^il) ,

**√i .** *Al*

where p is the integral part of *q* the integral part of 2^+2’ and *r* the integral part of —.

Sin $ *A,* cos *⅛A* are given in terms of sin *A* by the formulæ

2 sin *⅛A* = ( - l)p'(l + sin Λ)\* + ( - l)ρ'(l - sin Λ)⅛,

2 cos ⅜*A* = ( - 1∕,,(1 + sin ∠t)i - ( - l)i'(l - sin *A'f,*

*A* 1

where √ is the integral part of j>7r + 4 an<1 *t'* the integral part of

Λ 1

27γ 4‘

In any plane triangle *ABC* we will denote the lengths of the sides *BC, CA, AB* by *a, b, c* respectively, and the angles *BAC, ABC, ACB* by *A, B, C* respectively. The fact that the projections of *b* and c on a straight line perpendicular to the side *a* are equal to one another is expressed by the equation δsin *C=csinB* ; this equation and the one obtained by projecting c and *a* on a straight line perpendicular to *a* may be written . a .= -.—==-r‰. The r r j sin *A* sin *B* sin *C*

equation α=δ cos C,+c cosT? expresses the fact that the side *a* is equal to the sum of the projections of the sides *b* and *c* on itself ; thus we obtain the equations

*a-b* cos *C+e* cos

*b=c cosA + acosC }-.*

*c=a* cos *B + b cos A J*

1 f we multiply the first of these equations by - *a,* the second by *b,* and the third by c, and add the resulting equations, we obtain β2

the formula δ2 + c2-α2=2δc *cos A* or *cosA=* —-, which gives

the cosine of an angle in terms of the sides. From this expression for *cos A* the formulæ sin∣ *A =* —C- J^,cos⅛√4 = \*∣ '-^yy~

tan *⅛A = ■* ~~f~~~~\* ~j2ζ^~~ ~~f~~~~'~~ I sin j = ⅛∣ Ms - α)<s - δ)(s “ c)} i> where 5 denotes j(α + *b* + c), can be deduced by means of the dimidiary formula.

From any general relation between the sides and angles of a triangle other relations may be deduced by various methods of transformation, of which we give two examples.

(a) In any general relation between the sines and cosines of the angles *A, B,* O' of a triangle we may substitute *pA+qB + rC, rA +pB + qC, qA +rB+pC* for *A, B, C* respectively, where *p, q, r* are any quantities such that *p + q + r+1* is a positive or nega­tive multiple of 6, provided that we change the signs of all the sines. Suppose p + g, + r + l = *6n,* then the sum of the three angles 2α7r - *(pA +qB + rC), 2nπ - (rA +pB + qCl), 2nπ - (qA + rB +pC)* is *π* ; and, since the given relation follows from the condition *A + B* + C,= 7r, we may substitute for *A, B, C* respectively any angles of which the sum is 7r ; thus the transformation is admissible.

(∕3) It may easily be shown that the sides and angles of the triangle formed by joining the feet of the perpendiculars from the angular points *A, B, C* on the opposite sides of the triangle *ABC* are respectively *a* cos *A, b* cos *B, c* cos C, *π - 2A, π - 2B, π-2C∙,* we may therefore substitute these expressions for *a, b, c, A, B, G* re­spectively in any general formula. By drawing the perpendiculars of this second triangle and joining their feet as before, we obtain a triangle of which the sides are *-a cos A* cos 2 J, - δ cos 2? cos 22?, - c cos *Ceos 2C* and the angles are *4A - π, 4B - π, ½C - π ;* we may therefore substitute these expressions for the sides and angles of the original triangle ; for example, we obtain thus the formula

. . α2 cos2 *A* cos2 *2A — b∙* cos2 *B cos- 2B — c2* cos2 *C* cos2 *2C* COS 4 *A. ^^~\* - -*

*2bc* cos *B* cos *C cos 2B* cos *2C*

This transformation obviously admits of further extension.

(1) The three sides of a triangle *ABC* being given, the angles can be determined by the formula

Ztan- =10 +⅛log(s-δ) + ilog(s-c)-⅛logs-⅛log(s-a)

and two corresponding formulæ for the other angles.

(2) The two sides α, *b* and the included angle *C* being given, the angles *A, B* can be determined from the formulæ

*A+B=π-C,*

*L* tan *⅛(A - B)* = log *(a-b)-* log *(a + b) + L* cot ⅛C',

and the side *c* is then obtained from the formula

log c= log*a + B* sin *C-L* sin *A.*

(3) The two sides α, *b* and the angle *A* being given, the value of sin 2? may be found by means of the formula

Zsin27=Zsin∠i+logδ-logα ;

this gives two supplementary values of the angle *B,* if *b* sin *A < a.* If *bsinA>a* there is no solution, and if *b sin A —a* there is one solution. In the case *b* sin *A<a,* both values of *B* give solutions provided *b>a,* but the acute value only of *B* is admissible if *b<a.*

The other side *c* can then be determined as in case (2).

(4) If two angles *A, B* and a side *a* are given, the angle *C* is de­termined from the formula *C=τr - A- B* and the side *b* from the formula log *b =* log *a + L* sin *B - L* sin *A.*

The area of a triangle is half the product of a side into the per- j pendicular from the opposite angle on that side ; thus we obtain < the expressions *⅛bcsinA,* {s(s - *a)(s - b)(s* - c)} 1 for the area of a≈ triangle. A large collection of formulæ for the area of a triangle ≈ are given in the *Annals of Mathematics* for 1885 by Μ. Baker. <

Let *a, b, c, d* denote the lengths of the sides *AB, BC, CD, DA* 1 respectively of any plane quadrilateral and *A + C=* 2α ; we may obtain an expression for the area *S* of the quadrilateral in terms of the sides and the angle α.

We have *2S=ad sin A+ bc sin (2a-A)*

and ⅜(α2 + <Z2 - δ2 - c2) *= adcosA-bccos (2-a-A) ;*

hence *4S^2* + ⅛(α2 *+ d2-b2-* c2)2 = α2 c? + δ2 c2 - *2abcd* cos 2a.

If *2s=a + b + e + d,* the value of *S* may be written in the form

*S =* {s(s - *d)(s - b)(s -e')(s-d')- abed* cos2 α} L

Let *R* denote the radius of the circumscribed circle, *r* of the in­scribed, and *r1, r2, r3* of the escribed circles of a triangle *ABC ;* the values of these radii are given by the following formulæ.

τ>~αδε\_ *a* ∙κ~4S^2smΛ,

*S*

*r=-=(s-a')* tan⅜ *A = 4B* sin⅜ *A* sin⅜ *B* sin⅜ *C,*

*g*

*r1==s* tan∣ *A = !Rsin⅛A* cosj *B* cos⅜ *C.*

*Spherical Trigonometry.*

We shall throughout assume such elementary propositions in spherical geometry as are required for the purpose of the investiga­tion of formulæ given below.

A spherical triangle is the portion of the surface of a sphere bounded by three arcs of great circles of the sphere. If *BC, CA, AB* denote these arcs, the circular measure of the angles subtended by these arcs respectively at the centre of the sphere are the sides *a, b, c* of the spherical triangle *ABC* ; and, if the portions of planes passing through these arcs and the centre of the sphere be drawn, the angles between the portions of planes intersecting at *A, B, C* respectively are the angles *A, B, C of* the spherical triangle. It is not necessary to consider triangles in which a side is greater than 7r, since we may replace such a side by the remaining arc of the great circle to which it belongs. Since two great circles intersect each other in two points, there are eight triangles of which the sides are arcs of the same three great circles. If we consider one of these triangles *ABC* as the fundamental one, then one of the others is equal in all respects to *ABC,* and the remaining six have each one side equal to, or common with, a side of the triangle *ABC,* the opposite angle equal to the corresponding angle of *ABC,* and the other sides and angles supplementary to the corresponding sides and angles of *ABC.* These triangles may be called the associated triangles of the fundamental one *ABC.* It follows that from any general formula containing the sides and angles of a spherical triangle we may obtain other formulæ by replacing two sides and the two angles opposite to them by their supplements, the remain­ing side and the remaining angle being unaltered, for such formulæ are obtained by applying the given formulæ to the associated triangles.

If *A,,B,,C'* are those poles of the arcs *BC, CA, AB* respectively which lie upon the same sides of them as the opposite angles *A, B,C,* then the triangle *A'HC'* is called the polar triangle of the triangle *ABC.* The sides of the polar triangle are *π- A, π - B, π - C,* and the angles *τr - a, ττ -b, π — c.* Hence from any general formula connecting the sides and angles of a spherical triangle we may obtain another formula by changing each side into the supplement of the opposite angle and each angle into the supplement of the op­posite side.

Let *0* be the centre of the sphere on which is the spherical triangle *ABC.* Draw *AL* per­pendicular to *OC* and *AM* perpendicular to the plane *0BC.* Then the projection of *OA* on *OB* is the sum of the projections of *OL, LM, MA* on the same straight line. Since *AM* has no projection on any straight line in the plane *0BC,* this gives *OA* cos *c = CL* cos *a + LM* sin *a.*

Now *OL = CA cos b, LM=AL* cos *C= OA* sin δ cos *C ;*

therefore cos c= cos *a* cos δ + sin *a* sin δ cos *C.*

We may obtain similar formulæ by interchanging the letters *a, b, c,* thus cos *a*=cos δ cos *c* + sin δ sin *c* cos *A*1

cosδ = cose cosα +sine sinα cos2? 4 (1).

cosc=cosα cosδ + sinα sinδ cosC,*J*