These formulæ (1) may be regarded as the fundamental equations connecting the sides and angles of a spherical triangle ; all the other relations which we shall give below may be deduced analytically from them ; we shall, however, in most cases give independent proofs. By using the polar triangle transformation we have the formulæ cos *A =* - cos *B* cos *C+* sin *B* sin *C* cos *a* j cos Z?= - cos *C* cos *A* + siu *C* sin *A* cos *b* J- (2).

cos *C= -* cos *A* cos *B* + sin *A* sin *B* cos c J

In the figure we *have AM=ALsiιιC=rsinb* sin O', where r denotes the radius of the sphere. By drawing a perpendicular from *A* on *OB,* we may in a similar manner show that *AM= r* sin *c* sin *B,* therefore sin *B* sin c=sin *C* sin *b.*

By interchanging the sides we have the equation sin *A* \_sin \_Z?\_sin *C\_l·* sin *a ~* sin *b ~* sin *c ’*

we shall find below a symmetrical form for *k.*

If we eliminate cos *b* between the first two formulæ of (1) we have cos *a* sin2c = sin *b* sin *c cos A* + sin *c* cos *c* sin *a* cos *B ;*

therefore cot *a* sin c = cos *A* + cos c cos *B* sιn<ι

=sin *B* cot *A* + cos *c* cos *B.*

We thus have the six equations

cot *a* sin *b =* cot *A* sin *C+* cos *b* cos C'∣ cot δsinα=cot *B* sin C,+ cosα cos *C* I cot *b* sin c=cot *B* sin *A* + cos c *cos A* I ...

cot *c* sin δ = cot *CsinA* + cos *b cos A* j '4',

cot *c* sin *a* = cot *C* sin *B* + cos *a* cos *Β ∣ cota* sin *c = cotA* sin 2? + cos c *cosB J*

When *C=n* formula (1) gives , , ,

2 o cos c=cos *a cos b* (a),

and (3) gives sin δ = sin *B* sin *c ∣ .*

sin α = sin Λ sin c I ’

from ( 4) we get tan *a*=tan *A* sin *b* = tan *c* cos *B (*

tan δ=tan22 sinα = tan *c cosA )*

The formulæ cos *c — cotA* cot 7? (e)

and . *cos√4=cos asinB (*

*cosB=cosbsinA∖*

follow at once from (a), *(β),* (γ). These are the formulæ which are used for the solution of right-angled triangles. Napier gave mnemonical rules for remembering them.

The following proposition follows easily from the theorem in equation (3) :—If *AD, BE, GF* are three arcs drawn through *A, B,C* to meet the opposite sides in *D, E, F* respectively, and if these arcs pass through a point, the segments of the sides satisfy the relation sin *BD* sin *CE* sin *AF=* sin *CD* sin *AE*sin *BF∙,* and conversely if this relation is satisfied the arcs pass through a point. From this theorem it follows that the three perpendiculars from the angles on the opposite sides, the three bisectors of the angles, and the three arcs from the angles to the middle points of the opposite sides, each pass through a point.

If *D* be the point of intersection of the three bisectors of the angles *A, B, C,* and if *DE* be, drawn perpendicular to *BC,* it may be shown that *BE=* j(α + c- *b)* and *CE=⅛(a + b - c),* and that the angles *BDE, ADC* are supplementary. We have

1 sine sin *ADB* sinδ sin *ADC , , . A*

,ls° ≡Γb5=-—· s⅛cδ=-Η-i theref"rc s,",2

sin — sin —

2 2

sin *BD* sin *CD* sin *CDE* sin *BDE τ, . . \_ τ, . \_ r, \_ . „ „*

*= .—i*—τ . But sin *BD* sin *BDE=* sin *BE*

sin *b* sin c

=sin ~~+~~~~g~~——, and sin *CD* sin CZ)^=sin C'^=sin^-^—- ; there- Î. *a + c-b . a + b-c)\**

sin — sin—- ∕

-2 ■ -.-. -2-- ( (5)∙ sin *b* sin c ∖

Apply this formula to the associated triangle of which *π-A, π ~ B> C* are the angles and *π - a, π-b, c* are the sides ; we obtain *( . b+c-a . a + b + c) 1*

. ∖ sin—- sin —-— f

the formula cos — = ) 2 ( (6).

2 ∕ sin *b* sin c ∖

By division we have

*{. a + c-b . a+b-c∖⅛*

sin 2~ sιn 2~^^

r-= U-} (7),

*. b+c-a . a+b+c[* sin-^-mn-^—J and by multiplication

Â 2 *f . a + b + c . b+c-a . c + a-b . a + b-c^∣l* sin *b* sure I 2 2 2 2 J

~~=~~~~s~~~~i~~~~n~~ ~~⅞~~ ~~s~~~~∣~~~~n e~~ < 1 ~ cos2 a ~ cos2 & - cos2 c + 2 cos α cos δ cos c} 1∙

Hence the quantity *k* in (3) is

~~sinα sin δ sine~~*{1 “ co≡2a ~ c°s' b ~ co^c + 2∞\*n ∞≡ l>* cosc}\* (8). Apply the polar triangle transformation to the formulæ (5), (6), (7), (8) and we obtain

*„ ( A + C-B A+B-C∖i*

c0s2= y08 2 c°8 —2 [ WJ

t sin *B* sin *C l*

„ i *„„B+C-A A + B+C∖h* sina= I -COS — COS — I (IQ);

t sin *B* sin *C '*

*Î B + C-A A+B + Cλi*

-c°s 2 cos —

*A + C-B A + B-C∖ ^11)∙*

cos —2 C0S *2 )*

If = "∙"^sj11^siny {1 *~oos2A -cos2B-cos2C-2cosAcosBcosC}* 1, we have *kk'=l* (12).

Let *E* be the middle point of *AB* ; draw *ED* at right angles to *AB* to meet *AC in D ;* then *DE* bisects the angle *ADB.* Let *CF* bisect the angle *DCB* and draw *FG* perpendicular to *BC,* then

C(7=⅛, *^FBE = ½Α*

*^FCG=90o-* J

From the triangle *CFG* we have cos *CFG*

= cos *CG* sin *FCG,* and from the triangle

*FEB* cos *EFB*=cos *EB* sin *FBE.* Now

the angles *CFG, EFB* are each supplementary to the angle *DFB,* therefore

*a-b C . A + B ~ c*

cos — cos -2 = sin *—— cos* (13).

Also sin *CG=*sin *CF* sin *CFG* and sin *EB =* sin *BF* sin *EFB ;*

*.1 f . a-b C . A-B . c ,, t.*

therefore sin —cos - = sin —-— sin - (14).

Z 2 2 2

Apply the formulæ (13), (14) to the associated triangle of which *a,* 7r - δ, π - *c, A, π - B, π - C* are the sides and angles, we then have *. a + b . C A - B . c*

sin —2~ sin y = cos —7r- sin 2 (15),

*a + b . C A + B c*

cos —~ sin -g- = cos —g— cos - (16).

The four formulæ (13), (14), (15), (16) were first given by Delambre in the *Connaissance des Temps* for 1808. Formulæ equivalent to these were given by Mollweide in Zach’s *monatliche Correspondenz* for November 1808. They were also given by Gauss ( *Theoria motus,* 1809), and are usually called after him.

From the same figure we have

tan *FG—*tan *FCG* sin *CG—*tan *FBG* sin *BG ;*

*.. ?» , C . a — b A — B . a + b*

therefore cot - sin —=tan —— sin — ,

z z z z

*. a* - δ

*i A-B* sιn~2- iC,

or tan-r-=-—0c°t2 (17).

sin —

Apply this formula to the associated triangle *{π - a, b, π-c, τr- A, B,* 7τ *-C),* and we have

*a + b*

*,A+Bc°s~2~. C*

cot-=~S^4ta,,2'

∞s-5-

*a-b*

*A + B* c°s-2- *iC*

or tan- = —Γ+δc°t2 (18)·

cos-^-

If we apply these formulæ (17), (18) to the polar triangle, we have

*. A-B*

\_ sin —-—

tan⅛2 = -3^βtm4 (19);

sin-y-

. cos

*a + b* 2 i c

tanT= ^Γ+⅛til',2 <20>∙

c°5-2-

The formulæ (17), (18), (19), (20) are called Napier’s “Analogies”; they were given in the *Mirif. logar. canonis descriptio.*