*„„ , , . . a . b . c a b c .*

If we use the values of sin -, sin-, sin -, cos -, cos cos 2 given by (9), (10) and the analogous formulæ obtained by interchanging the letters, we obtain by multiplication

*. a b . „ . c B+C-A>*

sin g cos - sin *C=* sin cos -

*a* δ . zv c *A + B-C*

cos - cos - sin *C =* cos *r* cos θ I.

2 2 2 2

*. a . b . zy e A* 4^ *B + C*

sin 2 sin - sin *C=* cos cos —

These formulæ were given by Schrneisser in *Crelle,s Joum.,* vol. x.

The relation sin *b* sin *c* + cos *b* cos *c cos A* = sin# sin C’ - cos# cos C,cos α was given by Cagnoli iu his *Trigonometry* (1786), and was redis­covered by Cayley *{Phil. Mag.,* 1859). It follows from (1), (2), and (3) thus : the right-hand side of the equation equals sin *B* sin *G + cosa{cosA* - simβsinC'cosα)=sin^sinC'sin¾ + cosαcos∠4, and this is equal to sin *b* sin *e* + cos *A* (cos *a* - sin *b* sin *c* cos *A )* or sin *b* sin *c* 4- cos *b cos c* cos *A.*

The formulæ we have given are sufficient to determine three parts of a triangle when the other three parts are given ; moreover such formulæ may always be chosen as are adapted to logarithmic calcu­lation. The solutions will be unique except in the two cases (1) where two sides and the angle opposite one of them are the given parts, and (2) where two angles and the side opposite one of them are given.

Suppose *a, b, A* are the given parts. We determine *B* from the formula sin #=-JU sin *A* ; this gives two supplementary values of sin *a*

*B,* one acute and the other obtuse. Then (7 and *c* are determined from

*. a-b . A + B*

*, . C* suι ~2~ *A-B +* c sin~2-t *a-b*

the equations tan- = —∩ + ⅛cot—2—’ tano =—^ΓUβtan~2^^' sin∏r sin-T^

*C c*

Now tail 2» tan 2 must both be positive ; hence *A-B* and *a - b* must have the same sign. We shall distinguish three cases. First, suppose sin *b* < sin *a* ; then we have sin *B <* sin *A.* Hence *A* lies be­tween the two values of *B,* and therefore only one of these values is admissible, the acute or the obtuse value according as *a* is greater or less than *b* ; there is therefore in this case always one solution. Secondly, if sin *b* > sin *a,* there is no solution when sin *b* sin *A >* sin *a ;* but if sin *b* sin *A <* sin *a* there are two values of *B* both greater or both less than *A.* If *a* is acute, *a-b,* and therefore *A-B,* is negative ; hence there are two solutions if *A* is acute and none if *A* is obtuse. These two solutions fall together if sin *b* sin *A* = sin *a.* If *a* is obtuse there is no solution unless *A* is obtuse, and in that case there are two, which coincide as before if sin *b* sin *A* = sin *a.* Hence in this case there are two solutions if sin *b* sin *A* ≤,sin *a* and the two parts *A, a* are both acute or both obtuse, these being coinci­dent in case sin *b* sin *A =* sin *a* ; and there is no solut ion if one of the two *A, a* is acute and the other obtuse, or if sin *b* sin *A* > sin *a.* Thirdly, if sin *b =* sin *a* theu *B—A* or *π - A.* If *a* is acute, *a - b* is zero or negative, hence *A - B* is zero or negative ; thus there is no solution unless *A* is acute, and then there is one. Similarly, if *a* is obtuse, *A* must be so too in order that there may be a solution.

If *a = b=* 5, there is no solution unless *A = -,* and theu there are an infinite number of solutions, since the values of *C* and *c* become indeterminate.

The other case of ambiguity may be discussed in a similar manner, or the different cases may be deduced from the above by the use of the polar triangle transformation. The method of classification

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according to the three cases sinδ=sinα was given by Professoι Lloyd Tanner *{Messenger of Math.,* vol. xiv. ).

If *r* is the angular radius of the small circle inscribed in the *Λ*

l triangle *ABC,* we have at once tan r=tan y sin *{s-a),* where *2s=a + b + c ;* from this we can derive the formulæ

*NA B C . . B . C A ....*

tanr=n cosecs= -^sec^sec — sec 2^=sιnα sinsin— sec —(21), where *n, N* denote the expressions

{sin s sin (s - *a)* sin (s - 5) sin *{s -* c)} I,

{ - cos *Seos {S-A)* cos *{S-B)* cos *{S-* C,), »·

The escribed circles are the small circles inscribed in three ol the associated triangles ; thus, applying the above formulæ to the triangle *{a,π -b, ιr - c,A, π - B,π - C),* we have for r1, the radius ol the escribed circle opposite to the angle *A,* the following formulæ *A . . . N A B C*

tan r1=tan — sin *s=n* cosec *{s-a) =* ^-sec — cosec cosec

*• B C A . ..*

= sin *a* cos ∩ cossec θ (22).

The pole of the circle circumscribing a triangle is that of the circle inscribed in the polar triangle, and the radii of the two circles are complementary ; hence, if *R* be the radius of the circum­scribed circle of the triangle, and *Rx, R2, R3* the radii of the circles circumscribing the associated triangles, we have by writing *R*

*J* for *r, - -R1* for *r1, π-a* for *A,* &c., in the above formulæ

2 *a ,n ., u a b c .τ „* cot *R=*cot 2 c0s ~ = £ cosec 2 cosec 2 cosec *2 = sec*

*. b c a* = sin *A* cos 2 cos cosec (23) ;

*„ .ar,n a b c ∖τ ∕<y j∖*

cot *R1=* - cot - COS *S=*2 cosec 2 sec ö sec 2 ~ SeC - A '

*. . ■ b . c a ....*

— sin *A* sin - sin - cosec - (24).

The following relations follow from the formulæ just given :—

2 tan *R =* cot *r1* + cot r2+cot *r3 -* cot *r,*

2 tan,K1 = cotr + cotr2+cotr3-cotr1,

tan *r* tan r1 tan r2 tan rs=n2, sin2 s=cot *r* tan r1 tan r2 tan *r3,* sin2 *{s-a) =* tan *r* cot *r1* tan r2 tan *r3.*

*HE=A+B + C-τr,* it may be shown that *E* multiplied by the square of the radius is the area of the triangle. We give some of the more important expressions for the quantity *E,* which is called the spherical excess.

*A + B a + b . A + B a-b*

cos —2 cos -g- sin —2 - cos

We have ~— = and ——= »

*. C c C c*

sin 2 cos 2 cos cos

. ∕(7 *E∖ a + b (C E∖ a-b*

sm(--2) cos— j cos (2-2) cos ^ 2^^

or —~σ—=—rand *C~=*—Γ ;

sin g- cos 2 cos — cos

*. C . (C-E∖ c a + b*

sin — - sin —2~) cos 2 - cos ^^2~

*hence ~C . (C-E\= e a + b ,*

sin - + sin ( —2~ 1 cos + cos

*+ E*

tan-τ

*A g g Q*

therefore 7=—⅛= tan - tan ——

n o — *rj Δ A*

tan-—-—

Similarly tan j tan2 ^∙^- = tan -- tan - 2~ 5 therefore tan tan ∣ tan s-^ tan tan L~- L (25).

This formula was given by L’Huillier.

*a + b*

*.. . C E C . E* c0s-2~ . *C*

Also sin i√ cos 7τ - cos -x sin \_ = sin s ;

2 2 2 2 c 2 ’

cos 2

*C E . C . E c°s~2~ C*

cos2 C0S 2+sin2 sm 2 = c^c°s2 ’

cos 2

whence, solving for cos 2, we get

*E* 1 + cos α 4-cos 5 +cosc .na.

cos θ=- √ — (26).

2 *. a o c*

*4 cos* 2 cos 2 cos g

This formula was given by Euler *{Nova acta,* vol. x.). If we find sin— from this formula, we obtain after reduction

*. E n*

sin -s = √ .

2 \_ *a* 0 c

2 cos- cos θ cos -

Z *J Ji*

a formula given by Lescell *{Acia Petrop.,* 1782).

From the equations (21), (22), (23), (24) we obtain the following formulæ for the spherical excess :—

sin2 — = tan *R* cot *Ri* cot *R2* cot *R3*

*Ji*

4(cot r1 + cot r2 + cot ra)

- (cot *r -* cot *r1* + cot *r2* + cot *r3)* (cot *r* + cot r1 - cot r2 4- cot r3)' (cot *r* + cot r1 + cot r2 - cot r3)

The formula (26) may be expressed geometrically. Let *M, N* be the middle points of the sides *AB, AC.* Then we find cos JZA

1 + cos *a* + cos ⅛ + cosc , *E a*

= i : hence cos Ή = cos *MN* sec s.

*. b c ,* 2 2

4 cos - cos g

A geometrical construction has been given for *E* by Gudermann (in *Crelle's Journ.,* vi. and viii). It has been shown by Corneliu.- Keogh that the volume of the parallelepiped of which the radii ol