the sphere passing through the middle points of the sides of the triangle are edges is sin - .

Let *ABCD* be a spherical quadrilateral inscribed in a small circle ; let *a, b, c, d* denote the sides *AB, BC, CD, DA* respectively, and *χ, y* the diagonals *AC, BD.* It can easily be shown by joining the angular points of the quadrilateral to the pole of the circle that *A + C=B+D. If* we use the last expression in (23) for the radii of the circles circumscribing the triangles *BAD, BCD,* we have

*• a d y . „ b C y*

sin *A* cos X cos x cosec 3=sin *C* cos 3 cos θ cosec 3 ; *J J J <ii Λ Ji*

, sin *A* sin *C*

whence -, — -

*be ad* cos 3 cos 3 cos - cos 3 *J J J Ji*

This is the proposition corresponding to the relation *A + C=tτ* for a plane quadrilateral. Also we obtain in a similar manner the theorem

8i„? lhi

*• τ> · i d’*

sin *B* cos - sin *A* cos -

analogous to the theorem for a plane quadrilateral, that the diagonals are proportional to the sines of the angles opposite to them. Also the chords *AB, BC, CD, DA* are equal to 2 sin 3, 2 sin θ, 2 sin 3, 2 sin 3 Z Z Z Z respectively, and the plane quadrilateral formed by these chords is inscribed in the same circle as the spherical quadrilateral ; hence by Ptolemy’s theorem for a plane quadrilateral we obtain the analogous theorem for a spherical one

sin I sin ∣=sin " sin ∣+sin ∣ sin ∣.

It has been shown by Remy (in *Crelle's joum.,* vol. iii.) that for any quadrilateral, if z be the spherical distance between the middle points of the diagonals,

cos *a* + cos *b* + cos *c* + cos *d*=4 cos *⅛x* cos ⅛y cos ⅛z.

This theorem is analogous to the theorem for any plane quadri­lateral, that the sum of the squares of the sides is equal to the sum of the squares of the diagonals, together λvith twice the square on the straight line joining the middle points of the diagonals.

A theorem for a right-angled spherical triangle, analogous to the Pythagorean theorem, has been given by Gudermann (in *Crelle's journ.,* vol. xlii,).

*Analytical Trigonometry.*

Analytical trigonometry is that branch of mathematical analysis in which the analytical properties of the trigonometrical functions are investigated. These functions derive their importance in ana­lysis from the fact that they are the simplest singly periodic functions, and are therefore adapted to the representation of undu­lating magnitude. The sine, cosine, secant, and cosecant have the single real period 2τr ; *i.e.,* each is unaltered in value by the addi­tion of 2π to the variable. The tangent and cotangent have the period τr. The sine, tangent, cosecant, and cotangent belong to the class of odd functions ; that is, they change sign when the sign of the variable is changed. The cosine and secant are even func­tions, since they remain unaltered when the sign of the variable is reversed.

The theory of the trigonometrical functions is intimately con­nected with that of complex quantities,—that is, of quantities of the form *x + vy* (ι = ∖∕-l). Suppose we multiply together, by the rules of ordinary algebra, two such quantities, we have

(a⅛ + tZ∕ι)(¾ + \*y2) = (a⅛a¾ - 3∕ι2∕a) + \*(¾ya + ¾V1)∙.

We observe that the real part and the real factor of the imaginary part of the expression on the right-hand side of this equation are similar in form to the expressions which occur in the addition formulae for the cosine and sine of the sum of two angles ; in fact, if we put a¾=r1 cos 01, y1=r1sin01, a‰=r2cos02, w2=r2sin02, the above equation becomes ' 2 2,

r1(cos 01 +1 sin 01) × r2(cos *θ2* +1 sin 02)=r1r2(cos 01 + 02 +t sin *θ^+θ2).*

We may now, in accordance with the usual mode of representing complex quantities, give a geometrical interpretation of the meaning of this equation. Let *P1* be the point whose coordinates referred to rectangular axes *Ox, Oy* are *x1, y1 ;* then the point *P1* is employed to represent the quantity *x1 + ιy1.* In this mode of representation real quantities are measured along the axis of *x* and imaginary ones along the axis of *y,* additions being performed according to the parallelogram law. The points *A,A1* represent the magnitudes ±1, the points *a,a1* the magnitudes ±t. Let *P2* represent the expression a⅛ + \*⅜ an<1 p the expression (a¾ + √∕1)(a⅞ +\*2∕2)∙ The quantities *2'oθx,r2,θ2* are the polar coordinates of *P1* and *P2* respectively referred to *O* as origin and *Ox* as initial line ; the above equation shows that r1r2 and *θi + θ2* are the polar coordinates of *P ;* hence *OA : OP1 :: OP2 : OP* and the angle *POP2* is equal to the angle *PxOA.* Thus we have the following geometrical construc­tion for the determination of the point *P.* On *OP2* draw a triangle similar to the triangle *OAP1* so that the sides *OP2, OP* are homologous to the sides *OA, OP,* and so that the angle *POP2* is positive ; then the vertex *P* represents the product of the expressions represented by P1,Z2∙ If *x2 + ιy2* were to be divided by a¾ + ty1, the triangle *OPP2* would be drawn on the negative side of similar to the triangle *OAPx* and having the sides *OP, OP2* homologous to *OA, OP,* and *P* would represent the quotient.

If we extend the above to *n* complex quantities by continual repeti­tion of a similar operation, we have—

(cos 01 +*1* sin 01) (cos *θ2* +t sin 02)...

(cos *θn* +1 sin *θn)*

= cos(01 + 02+... + 0,1) + tsin(01 + 02 + ...

+ 0∏)∙

If 0j = 02= ... *= θn=* 01, this equation becomes (cos 0 + ιsin 0)" = cos *nθ* +1 sin *nθ ;* this shows that cos 0 +*ι* sinØ is a value of (cosnØ + 1 $ *θ θ*

tsinnØ)«. If now we change 0 into -, we see that cos- + t sin - is a

1 *η η n*

value of (cosØ + tsinØ)« ; raising each of these quantities to any positive integral power *m,* cos^-f-tsin- is one value of (cos 0 *m η n*

+ ιsin0)n. Also

*∕ m∖. . ∕ mA* 1

cost J0 + tsιn( —0 1 = ·

*∖ nj ∖ n ∕ ma,∙ma*

cos—0 + tsnι — 0 *η n*

hence the expression of the left-hand side is one value of 1

z~ —

(cosø + t sin 0)m''lorof (cos 0+ tsin0)^n. We have thus De Moivre’s theorem that cos *kθ* +ι sin *kθ* is always one value of (cos 0 +1 sin 0)i, where *k* is any real quantity.

The principal object of De Moivre’s theorem is to enable us to *vι*

find all the values of an expression of the form (α + d>)", where rn and *n* are positive integers prime to each other. If α=rcos0, fn *m*

*b = r* sin 0, we require the values of *rn* (cos 0 +1 sin 0) n. One value is immediately furnished by the theorem ; but we observe that, since the expression cosØ + tsinØ is unaltered by adding any multiple of *n , a .1 n,, r* w.0 + 2s7γ . ∞.0 + 2s7rλ. ,

2∙π∙ to 0, the -th power of *rn* ( cos H sin hs *a + w,*

*m 1 ∖ η n ∕*

if s is any integer ; hence this expression is one of the values re­quired. Suppose that for two values s1 and s2 of *s* the values of this

,, .1 ., rn.0 + 2s17r w.0 + 2s2tγj

expression are the same ; then we must have i —

*n 11*

a multiple of *2π* or s1 - *s2* must be a multiple of *n.* Therefore, if we give *s* the values 0,1, 2, . . . *η-* 1 successively, we shall get *n* differ- *m*

ent values of (α + *ιb)n,* and these will be repeated if we give *s* other ♦n

values; hence all the values of *(a+ιb)n* are obtained by giving *s* the values 0, 1, 2, . . . *n-1* in the expression *r"* (cos w, ^l^

*, . rn.θ + 2sπ∖ . , „ , . , b*

+ 4 sin — L where *r=*(α2 + *b2)i* and 0=arc tan - ,

We now return to the geometrical representation of the complex quantities. If the points *B1, B2, B3,... Bn* repre­sent the expression *x + ιy,{x + ιy')2, (x +ιy)3,... (x+ιy}n* respectively, the triangles *OAB1, OB1B2,... OBn~1Bn* are all similar. Let *(x+ιy)n=a + ιb,* then the con verse problem of finding the nth root of *a + ιb* is equivalent to the geometrical problem of describ­ing such a series of triangles that *OA* is the first side of the first triangle and *OBn* the second side of the nth. Nowit is obvious that this geometrical problem has more solutions than one, since any number of com­plete revolutions round *O* may be made in travelling from *By* to *Bn.* The first solution is that in which the vertical angle of each triangle is *—B OA* ; the second is that in which each is in

*η n*

this case one complete revolution being made round *O ;* the third