lias - *CBnCA* + 4τr) for the vertical angle of each triangle ; and so *n*

on. There are *n* sets of triangles which satisfy the required condi­tions. For simplicity we will take the case of the determination of the values of (cos *θ +1* sin 0)⅛. Suppose *B* to re­present the expression cos *θ +ι* sin *θ.l* If the angle *AOP1* is ⅜0, *P1* represents

*θ θ*

the root cos - 4-4 sin - ; the angle *AOB u o*

is filled up by the angles of the three similar triangles *AOP1, P1Op1, pf)B.* Also, if *P3, P3* be such that the angles *P1OP2, P1OP3* are y”, y respectively, the two sets of triangles *AOPi, P3Op3, p3OB* and *AOP3, P3Oρ2, P<flB* satisfy the conditions of simi­larity and of having *OA, OB* for the bounding sides ; thus P2,

, 0 + 27γ . 0 + 2π∙ 0 + 4τr . 0 4-47γ

*P3* represent the roots cos —θ 11 sin —θ—, cos — h1 sin —g—

respectively. If *B* coincides with *A,* the problem is reduced to that of finding the three cube roots of unity. One will be repre­sented by *A* and the others by the two angular points of an equi­lateral triangle, with *A* as one angular point, inscribed in the circle.

The problem of determining the values of the nth roots of unity is equivalent to the geometrical problem of inscribing a regular polygon of *n* sides in a circle. Gauss has shown in his *Disquisi­tiones arithmeticæ* that this can always be done by the compass and ruler only when *n* is a prime of the form 2p +1. The determina­tion of the nth root of any complex quantity requires in addition, lor its geometrical solution, the division of an angle into n equal parts.

We are now in a position to factorize an expression of the form

\_ (α Ψ t⅛). Using the values which we have obtained above for ι

(α + ιδ)n, we have

s=7l^1Γ *θ + 2sιτ* . 04-2sτr∖Ι

*xn* - (α + *ιb) = P* æ-r (cos Hsιn ) ...(1).

s=0 L ∖ *n n ∕A*

If t = 0, α=l, this becomes

s=τt-l∣- 2s7Γ . 2s7γ^1

*xn- l — P* æ - cos 4 sin —- I

s=0 L *η n* J

**-- 1**

z , ,, iX÷Γ2 ∕ 2S7Γ , . 2sπ∙∖

= (æ-l)(sc+l)P (je-cos-y ±ι sin — *J*

= (as-l)(αι+l)P “ (z2-2rcos-y+ l)(neven)(2).

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xπ-l = (x-l)P 2 (x2- *2x* cos +1 ) (n odd) ...(3).

If in (1) we put a= - 1, *b — Q,* and therefore *θ-ιτ,* we have

„ 1 sn,l^1Γ 2s + lτr . 2s+lτrΙ

*xn* +1 = *P* æ - cos *1* sin 1

**s=o L** *n* **nJ**

71-2

*= P* ~ Γir2 - 2≈ cos ~~^~~~~s+~~ ~~l~~~~,r~~ +1~] (n even) (4).

s=0 L J

\_ n - 3

x" + l = (z + l)P 2 Γ¾2 - 2# cos 2s + +1~∣ (n odd) (5).

3=0 L nJ

Also *x2n - 2xnyn* cos *nθ+y2n*

*= (x,n — yn* cos *nθ* + 4 sin *nθ){afl - yn* cos *nθ -* 4 sin *nθ) s=n~1h θ + 2sιr, . θ + 2sπ∖*

*— P (x-ycοs* ±4s1n — I

s=0 ∖ η n ∕

s=n-l1- · s~ -1

*= P* 0 [≈≡-2a7∕cos0+-+y2J (6).

Airy and Adams have given proofs of this theorem which do not involve the use of the symbol 4 (see *Camb. Phil. Trans.,* vol. xi.).

A large number of interesting theorems may be derived from De Moivre’s theorem and the factorizations which we have deduced from it ; we shall notice one of them.

In equation (6) put 3∕=-> take logarithms, and then differentiate each side with respect to *x,* and we get

2n(a2w~1 - ar2n^1) \_ ς= n -1 2(z - *x~s)*

*x"n -* 2 cos *nθ + x~2n~ s \_ 0 ~ ~ ~ 2sπ ~*

*s~υ x2-2cοs0 + -—* + sb~2

*a n*

Γutic2=y, then we have the expression

n(α2" - *b2n) (a2 — b2)* (α2n — *2anbn* cos *nθ* 4- *b2n)* for the sum of the series

**3 = 74-1 1**

ς =Ε7.

s-° a2-2aδcos04 t-δ'2

*n*

We shall now consider what meaning can be assigned to the symbol *dc+ιy.* The quantity *e* is defined as the limit of (1 + ^Y > where *n* is a positive quantity, and is increased indefinitely ; then, ∕ 1∖71Λ≈ ∕ *% ∖m*

for a real value of *x, ex* is the limit 0f(l + -1 or of (14— I , *∖ nJ ∖ mJ ’*

where *m=nx,* when *m* is increased indefinitely. We may define

**l i ∕ æ \_i\_** *ιty∖ Til*

*ex+iy* as the limit of (1 4—y—J when *m* is increased indefinitely. To determine the value of this limit put 1 +—*=r* cos *θ,-=r* sin *θ : rn m*

then e≈+i2' fs the limit of rw(cos∞04-4 *sinmθ),* and *rnι* is equal to

1 + 2 0r ultimately to (1 + 2j which has *cx* for

its limiting value. Also *θ* is arc tan —-— or —-— in the limit ;

*o x+m x+m*

hence *mθ* is ultimately equal to *y,* and thus the equation β≈+t3∕=ex(cos *y + ι* sin *y)* follows from our definition. It may be shown at once that × e\*ι+l3h = ex+xι+i(3∕+3∕ι) and, if we suppose that *ax+iy* denotes ∕z÷1^lo8α, we may show that complex expon­ents defined thus obey the same laws as real ones.

When the exponent is entirely imaginary we have, in accordance with the above definition,

*e·y=*cos *y +1* sin *y* and *e ~ ,y=*cos ( - *y)* + 4 sin( - *y) =* cos *y* - 4 sin *y ;* we thus obtain the exponential values of the sine and cosine— sin *y=- e~,y),eοsy=J(efy+e~,y).*

If we give imaginary or complex values to the variables in alge­braical expansions we obtain analogous trigonometrical theorems ; it is, however, necessary to consider the convergency of the series so obtained in order to determine within what limits the values of the variables must lie. If we expand *e,y* and *e~,y* by putting *ιy y3* and - *ιy* in the senes 1 *+y+ f~2 + j~~j 2~~* ~~3~~~~+~~ ~~'~~ ~~,~~

**• · 7∕δ** *Ιp*

we obtain the series sin *y = y - ⅛- + r⅞— r⅛- + ... ;*

I ö I 5 17

**,,2 ,4 1,6**

cos *y-*1 - jy- + ∣J- - βf + · · · ·

These series are convergent for all finite values of *y.* They may also be got from the expressions which we have obtained for the cosine and sine of a multiple of an angle in terms of the cosine and sine of the angle, and would thus be made to rest upon a basis independent of the symbol 4.

Consider the binomial theorem (α+δ)"=αn+nαn~⅛+^⅛T)αn-2j2+ \_ \_ <

L£

*n(n-l)... ln-r+l) n.r .*

4— i *-an~rV+ ... +bn.*

**∣-L**

Putting *a*=et9, *b=e~iθ,* we obtain

(2 cos *θ)n* = 2 cos *nθ + n2* cos *n - 2θ* 4- 2 cos it - 40 4- ...

t⅛A∙⅜"∙÷ 1) ct⅞ \_ 2ry+ . ..

When *n* is odd the last term is ~~'~~~~, ,~~ ~~-' '~~~~,1~~ ~~-~~~~ι~~~~i .∣ .~~

**∣⅜(^-D**

and when *n* is even it is

If we put α = etβ, *b= -e~lθ,* we obtain the formula *n*

( -1 )¾2 sin *θ)n =* 2 cos *nθ - 2n* cos(n - 2)0 4- ,\*⅛ y^2 cos(n - 4)0 - ...

when *n* is even, and

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(-1) 2 (2sin0)n=2sinzι0-π.2sin(n-2)04-~y-g⅛sin(7i-4)0...

**A- ¾(⅞-l),..⅜(π + 3)**

+ t } I ⅜(⅞-l)

when *n* is odd. These formulæ enable us to express any positive integral power of the sine or cosine in terms of sines or cosines of multiples of the argument. There are corresponding formulæ when *n* is not a positive integer.