Consider the identity log(l *-px)* + log(l-g∞) = log(l *-p + qx+pqxF).* Expand both sides of this equation in powers of *x,* and equate the coefficients of *x,l,* we then get

jp" + ⅛" = (? + *q)n ~ n{ρ* + ff)n~⅛ + ’ γy-3½> + *qiu~ip-f +...*

, „ .n(n-r-l)(n-r-2)... (n-2r + l)

+ (- 1 ) Δ -∣y2 i zιp + *gyι-rHgr + ...*

If we write this series in the reverse order, we have r+r=2(u)‰⅛ tai)S ’ ' (ω)s+⅛">(wΓ2f+’)‘

-⅛⅛PtaΓ,(ω),÷...÷(->)⅛÷tf] when *n* is even, and

p-+r=2( - ιΓ⅛√r (ψ) - ⅛⅛3ι^,(e∣-,)1

when *n* is odd. If in these three formulæ we put *p = elθ, q = e~iθ,* we obtain the following series for cos *nθ :—*

2 cos *nθ* = (2 cos *θ)n* - n(2 cos 0)"~2 + (2 cos *θ)n~i -...*

+( - ιr⅛>->∙-υ("->--¾-(>>.~⅛+υ(2c0^^+...(,) when *n* is any positive integer ;

*⅛l n2 „ n2(n2-2-) , n2(n2 - 22)(n2* - 42) β.

( - 1 )2 cos *nθ =* 1 - ∣-g- cos-0 + — cos40 -——∣-θ cosβ0

»

+ ... + (-l)22""1cosn0 (8)

when *n* is an even positive integer ;

*⅛ n{n2-l2) n(n2-l2)(n2-32) 5a*

(-1) 2 *cosnθ=nco8nθ - ∣* cos30÷ Γg cθs

*n* -1

.. .+(-1) f2"-1cos"0 (9)

when *n* is odd. If in the same three formulæ we put *p=eiθ, q= -e~iθ,* we obtain the following four formulæ :—

*n*

( - 1)22 cosn0=(2 sin 0)"- n(2 sin 0)"-2 + —9 3^(2 sin 0)"~i - ...

+ ( - l)r ~~∙~~~~1~~ ~~y ”~~~ ~~2r~~ ~~~~~~~1~~ ~~)~~(2 sin 0)"~2\*∙ + ... (n even)(10) ;

n-1

( - 1) 2 2 sin >ι0 = the same series (n odd) (11) ;

cos »«= 1 -1⅝ SÜP ⅛ + ⅛≤ srn∙ « - ~~⅜~~~~,~~~~-y-∙>~~~~i~~~~)~~ ~~5in~~~~. ,~~

+ ... + 2n-1 sin ,'0 (71 even) (12) ;

sin »«=» sin « - ,lt,⅛-1,-> sin» β+⅛W∑≤ βin≈ » - ...

I 3 I 5

71 - 1

+ ( - 1) 2 2"~1 sin" 0 (n odd) (13).

Next consider the identity *~~———=-—J ,.~~*

*∖-px ∖-qx l-{ρ + q)x+ρqχ-*

Expand both sides of this equation in powers of *x,* and equate the coefficients of x∙n-1, then we obtain the equation

*= (P* + S')"-1 - (n~2)(*p + q^)n~3pq +* ~~(~~~~h~~~~Σ~~~~3~~~~X^~~~~~4~~~~)~~(j, + ?)n-y3,2 \_ t φ e 2' 2 I

1λr (n-r-l)(n-7∙-2)...(n-2r), , „ .

+ (-l) p—i - *,(p + q)n-2r-Yqr+ ...*

If, as before, we write this in the reverse order, we have the series ( - i)b1[ 11('ψ)te)b'

÷≡≥t^3(42)W-+...+(-i)V,⅛÷fH] when *n* is even, and

n-lr- "-l „ „ „ n-3

( - ip [(w)^τ - n⅛(ψ) ⅛)τ

+ +(^+,r√∣ when *n* is odd.

If we put7?=e£0, *q=e~lθ,* we obtain the formulæ

sin?tØ=sinØ (2cos0)"-1-(n-2)(2cos0)"-3 + -—~∙^—-)(2cos0)"^s +( -1 r -^∙P^⅛cos<,)^-.+...} (14),

where 74 is any positive integer ;

*n .*

! ι∖2-1∙ *a ■ ai a* m(h2~22) 3zj 7((h2-22)(7t2-42) (-1) sιnjι0=sιn01 ncosØ- i-5—cos30+ -cos50-...

l I— L\_\_

--1

+ (-l)2 (2cos0)"~1 (71 even) (15) ;

( - 1) 2 sin «0 = sin 0 j 1 - ⅛p\*cθ8⅛ +^ 1^^^' ~ ¾os40 - ...

--ι 1

+ (-l)2 (2cos0)"~1 y (n odd) (16).

Ifwe put in the same three formulae j7=etβ, *q= -e~lθ,* we obtain the series

*n-2*

(-1) 2 sinntf = costf^sin\*~1i-(»-2)sinB-». + ^—i'l⅛in',~''7-... +(-ιr ~~("∑r-~~~~1~~~~)("-j∙-¾∙∙P-⅛~~->-⅛ +. J(„ evenj (17).

1

( - 1) 2 cos *nθ =* the same series *{n* odd) (18) ;

a o J ■ a *n(u2 -* 22) . ,z, π(n2 - 22)(it2 - 42) . .λ sin *nθ =* cos 0 1 *n* sin 0 ——r-—'sιn30 + - r⅛ sιn50 +

( I 3 I 4

- 1

... + ( - 1)2 (2 sin 0)n~1 J (neven) (19) ;

cos *nθ*=cos 011 - ,-yTsin20 + ~~3~~~~Λ~~in40 -...

+ (2 sin 0)"^1 j (71 odd} (20).

We have thus obtained formulæ for cos *nθ* and sinnØ both in ascending and in descending powers of cos 0 and sin 0. Viète ob­tained formulæ for chords of multiple arcs in powers of chords of the simple or complementary arcs equivalent to the formulæ (13) and (19) above. These are contained in his work *Theoremata ael angulares sectiones.* James Bernoulli found formulæ equivalent to (12) and (13) (*Mem. de l’Academie des Sciences,* 1702), and trans­formed these series into a form equivalent to (10) and (11). John Bernoulli published in the *Acta eruditorum* for 1701, among other formulæ already found by Viète, one equivalent to (17). These formulæ have been extended to cases in which *n* is fractional, nega­tive, or irrational ; see a paper by D. F. Gregory in *Camb. Math, joum.,* vol. iv., in which the series for cos *nθ,* siniiØ in ascending powers of cos 0 and sin 0 are extended to the case of a fractional value of τι. These series have been considered by Euler in a memoir in the *Nova acta,* vol. ix., by Lagrange in his *Calcul des fonctions* (1806), and by Poinsot in *Recherches sur l'analyse des sec­tions angulaires* (1825).

The general definition of Napierian logarithms is that, if *= a + ιb,* then *x + ιy =* log (α + *ιb).* Now we know that *ef+ly=efcοs y* + texsinyj hence σe cos *y=a, cc,* sin *y=b,* or ex=(α2 + 52)i, *y=* arc tan -±77iτ, where *m* is an integer. If *b—Q,* then *m* must be even or odd according as *a* is positive or negative ; hence

logc (α + ‘δ) = loge *{u2* + 02)^ + 4 (arc tan - ± *2nττ) CL*

or loge *{a + ιb') =* loge (α2 + *b2}⅛* + » (arc tan ± 2?i + π),

according as *a* is positive or negative. Thus the logarithm of any complex or real quantity is a multiple-valued function, the differ­ence between successive values being 2π4 ; in particular, the most general form of the logarithm of a real positive quantity is obtained by adding positive or negative multiples of 2π4 to the arithmetical logarithm. On this subject, see De Morgan’s *Trigonometry and Double Algebra,* chap, iv., and a paper by Prof. Cayley in vol. ii. of *Proc. London Math. Soc.*

λVe may suppose the exponential values of the sine and cosine extended to the case of complex arguments ; thus we accept e4(x+4⅛) + e - 4(x+4⅛) e4(x+43∕) \_ e - 4(x+4⅛)

ö and as the definitions of the *ί 2ι*

functions cos *(x+ιy),* sin *(x+ιy)* respectively. If æ=0, we have *eV + e-y , egA-p-y*

cos 47/=— and sin 4y=θ(e^-e y). The quantities —-—

*øy-e-y*

—2— are called the hyperbolic cosine and sine of *y* and are written cosh *y,* sinh *y* ; thus cosh ?/= cos ιy, sinh y = *—1* sin *ιy.* The functions cosh *y,* sinh y are connected with the rectangular hyperbola in a manner analogous to that in which the cosine and sine are