connected with the circle. We may easily show from the definitions that

cos2(x + *nf)* + sin2(z + ιy) = 1, cosh2 *y -* sinh2 *y =* 1 ;

cos(α∙ + *ιy)* = cos *x* cosh *y -ι* sin *x* sinh *y,* sin(x+*ιy) —* sin *x* cosh *y* +ι cos *x* sinh *y,* cosh(α *+ β) =* cosh *a* cosh *β* + sinh a sinh *β,* sinh(a *+ β)=* sinh a cosh *β -* cosh a sinh *β.*

These formulae are the basis of a complete hyperbolic trigonometry. The connexion of these functions with the hyperbola was first pointed out by Lambert.

If we equate the coefficients of *n* on both sides of equation (13), we get . zι 1 sin30 1.3 sin50 1.3.5 sin70 .n,,

Ø-sinØ + g 3+2.4 5 + 2.4.6 7 +

*θ* must lie between the values ± J. This equation may also be written in the form , χ3 j , χ, , 6 a,

are≡≈=≈+23 +0 5 +2TΓβ 7 +-

when *x* lies between ±1.

By equating the coefficients of *n2* on both sides of equation (12) veget 2 sin40 2.4 sins0 2.4.6 sin8#

9∙=sm≡β + - -3→3^ - r+37s77 - 4 + (22),

which may also be written in the form

*i · ∖ι ....∙>* 2 αr4 2.4 a:6 2.4.6 ≈8

(aresιn≈)'=^ + 3 2 +375 3 +375Γ7τ + -.

when *x* is between ±1. Differentiating this equation with regard to *x,* we get

arc sin ¾ 2, 2.4, 2.4.6 7

√l-a\*2^^x + 3x +3.5a^ + 3.5.7x +'"'

if we put arc sin *x —* arc tan *y,* this equation becomes aretani,=1-∣p∣l+∣1-⅛i + ^(r^y + ...] (23).

This equation was given with two proofs by Euler in the *Nova acta* for 1793. . . , , 7

τττ l 11 1 + *x* x3 z5 z7

"δl'ave -log—=χ+?+\_+γ + ... ;

put *ιy* for *x,* the left side then becomes ⅛{log(l + ιy)-log(l -ιy)} or ιarc tany±iii7r ; ,357

hence arc tany÷ιwr=y-^+^ - 4-....

The series is convergent if *y* lies between ± 1 ; if we suppose arc tan *y* restricted to values between ±-, we have

4 arctany=y-^ + ^- (24),

which is Gregory’s series. 0 &

Various series derived from (24) have been employed to calculate the value of ιr. At the end of the 17th century *π* was calculated to 72 places of decimals by Abraham Sharp, by means of the 7Γ 1

series obtained by putting arctanw=-, *y=-7=* in (24). The cal-

6 √3

culation is to be found in Sherwin’s *Mathematical Tables* (1742). About the same time Machin employed the series obtained from the equation 4 arc tan -arctanfT=y to calculate τr to 100 de- cimal places. Long afterwards Euler employed the series obtained 7r 1 1

from -=arc tan-+ arc tan-, which, however, gives less rapidly con­verging series (Introd., *Anal, in fin.,* vol. i. ). Lagny employed the formula arc tan—∕∣~θ to calculate τr to 127 places ; the result was communicated to the Paris Academy in 1719. Vega calculated *π* to 140 decimal places by means of the series obtained from the

τr 1 3 Ί

equation .- = 5 arc tan - + 2 arc tan -⅛ The formula ~ = arc tan „ +

1 4 J 7 10 4 2

arc tan ≈+ arc tan - was used by Dase to calculate *τr* to 200 decimal

Do

places. Rutherford used the equation *τr=*4 arc tan = - arc tan + 1 5/0

arctan99∙ 1 ,

If in (23) we put *y- -* and we have

o ∕

τ = 8arctail∣ + 4arctan) = 2,4 jl+∣.⅛ + ≡r\* Λ+...}

, cβ ( , 2 2 2.4/ 2 ∖2 }

+ 56P+3'100 + 3.5∖10θ) +,∙∙(,

a rapidly convergent series for *π* which was first given by Hutton in *Phil. Trans,* for 1776, and afterwards by Euler in *Nova acta* for 1793. Eider gives an equation deduced in the same manner from 1 3

the identity *π =* 20 arc tan ? + 8 arc tan ⅛-. The calculation of π has been carried out to 707 places of decimals ; see *Proc. Roy. Soc.,* xxi. and xxii. ; also Squaring the Circle (vol. xxii. p. 435 *sq.).*

We shall now obtain expressions for sin a; and cos a: as infinite products of rational factors. We have

*„ . x . x+π -9 . x . x + ιr . x + 2τr . x + 8τr*

sιnic=2sιn δ sin —=23 sin - sin —— sin 1 — sin — ;

Z Z 4 4 4 4

proceeding continually in this way with each factor, w-e obtain

*. 1 . X . x + π . x+2π . x+n-lπ*

sιnx=2n-1 sin - sin sin ... sin — -

*η η n u ’*

where *n* is any positive integral power of 2. Now

. ic + r7r . z + 7i-r7r *. x + rπ . rπ-x . n rπ* „x

sin —-— sin =sιn —-— sin =sιn2 sin2-,

*n n η η n u'*

*. . x + hnιr χ*

and sin — = cos -.

*η n*

Hence the above may be written

. ∩n-ι · *x ( .* o τr .ox∖∕.o2τr . oικ∖ sιnx=2 sin -( sιn2 — sin-- ) ( sιn2 sιn2 - 1...

»X *n n∕∖ n nJ*

*∕ . 9kπ . 9x∖ x*

I sin- sin2 - 1 cos -,

X *η nJ n*

where *k=⅛n-* 1. Let *x* be indefinitely small, then we have

2', 1 . *, π . ο 2π . 9kττ*

1 = sιn2 - sιn2 — ... sιn2 — ;

*η η η n*

hence

*• ~ . x x(7 sin2x∕n∖∕.* sin2 *x∣n∖* A sin2x∕n∖

*n n∖* sιn\*τr∕7t∕∖ *sιu22π∕nj ∖ sιu2kπnj*

We may write this

*■ . x xi7* sin2x∕π∖ A sin2z∕n ∖ τ,

sιnx=nsιn- cos - ( 1 - .- ,. I... ( 1 -*—l-r ]R, n n∖* s1n27r∕7i∕ ∖ *sιn2mπ∣nj*

where *R* denotes the product

Λ.J≡Lγ,.∠≤ ∖√.--f∖ ∖. -∙5ι1∙Λ -⅛-√ V -∙⅞√ and *m* is any fixed integer independent of *n.* It is necessary, when we make *n* infinite, to determine the limiting value of the quantity *R* ; then, since the limit of is and that of

sinwir/n. ., , ns^n«cosz

-mπ∣ll' 13 unιt∑> we have

⅛i÷SXι-⅛)∙∙∙0-i⅛>-

^Now *R* is less than unity, since sin - is less than sin ~~^t-.T~~ sin 7n + 2τr n n

-—... ; also by an elementar}7 algebraical proposition *R* is greatc r w -t · o≈ ( *9m + lπ „ kπ∖ π*

than 1 - sin -I cosec — l· ... + cosec- — ) and cosec 0<iτ-,if

*π n∖ n nJ 2θ*

Ö < - ; *R* is therefore greater than

1 a2/ 1 l 1 JLX

4Xm + l∣s ^+2∣a "\*+λ2Λ or than l-⅛i 1—I— —f∙ . +—1 11

4 l«i *m + 1 m + 1 m + 2 k-1 kΓ*

*x2 θx2*

or than 1 - —. Hence *R* = 1 - — , where *θ* is some proper fraction ; whence

si°≈=≈(1-i)(1-⅛)∙∙∙(ι-⅛)(ι-⅛)∙

When *m* is indefinitely increased this becomes

ri„ x=x(l -^)(1 - ⅛),. .=xF÷2(l ÷⅛) (25).

The expression for cos *x* in factors may be found in a simdar manner by means of the equation cosaj = 2sin —-^cos^1r or maybe deduced thus i

*i* 4a≈2 ∖

cos≈=sin2\*= × = 6 4χ2Vι\_ 4χ,2Wι\_4≤1

2sinz *pé* (1 π2)v 3⅛2)∖1 5⅛2)"∙

X *n2π2J*

*n=+∞∕ ‰ ∖*

=P (26).

n=-∞∖ 2ii + lτr∕

If we change *x* into *ιx,* we have the formulae for sinh *x,* cosh *x* as infinite products—

71=00 ∕ l>∙2 ∖ 7l=∞ ∕ *Aγ2 ∖*

sinhæ=æP ( 1+-i-A coshx=P ( 1 + -—).

*n=Q* X *n-\*H n=0∖* 2n + l∣2τr2∕

In the formula for sin a; as an infinite product put *x-^,* we then get 1 = . ~~2~~ ~~4 θ ‘~~ » if we st°P after 2n factors in the numer­

ator and denominator, we obtain the approximate equation

π 12.38.5a. ..(2⅞-l)2

1-2 22.42.62... (2n)2 ∙<2n + 1)