or ~~2~~~~'o κ\*\* o~~ "~ι = Λ^ίΜΓ> where *n* is a large integer. This ex- 1.3.5... ∙l

pression was obtained in a quite different manner by Wallis *{Arith­metica iv finitorum,* vol. i. of *Opp.}.*

wehave . , . (χ+,)√l+^)

sin(¾+y) \_ ∖ τtπ ∕

sinx ~ a√l+-≡-)

*∖ nπ ∕*

or cos *y* + sin *y* cot *x*

-(»♦SO ÷⅛)(1 ÷ΛX1 ÷⅛)(1 ÷⅛>∙∙∙ Equating the coefficients of the first power of *y* on both sides we obtain the series

cotx=l÷T- + -1—+ -⅛- +~~ς-∙⅜+~~ (27).

iBx + 7rie-7ra! + 2ιrx-27r

From this we may deduce a corresponding series for cosec *x,* for, since cosec *x=*cot '∣ - cot *x,* we obtain

111 1,1 1 1 . ∕0oλ

cosec*x=* l· , n- 4 ö Γ5—~~Q~ + ∙∙∙tz°λ

*x x + x X--ιc* a!+27r z-27Γ *x + οτr χ-ύπ*

By resolving J∩to factors we should obtain in a similar

o cos *x*

manner the series

, \_ 2 2 2 2 2 2 f291

anz-7γ-2x π + 2x 37γ - 2îc 3τr + 2ic 5τr-2x 07γ + 2x

and thence *pπ x∖* 2 2 2

sec *x=*tan(j + - tan ~ g~⅛

?—+ (30).

These four formulæ may also be derived from the product formulæ for sin *x* and cos *x* by taking logarithms and then differentiating.

, . 1 , cos a:

Glaisher has proved them by resolving the expressions tor and g∙^~∙∙∙ as products into partial fractions (see *Quark Journ. Math.,* vol. xvii.). The series for cot® may also be obtained by a continued use of the equation cotz = cot^ + co^—*2~* 7 ^see a PaPer

by Dr Schröter in Schlömilch’s *Zeitschrift,* vol. xiii.).

Various series for 7r may be derived from the series (27), (28), (29), (30), and from the series obtained by differentiating them one or more times. For example, in the formulæ (27) and (28), by putting τr

*x—* - we get

*n o*

ιr f, 1 1 1 1 1

7r = n tan -11 = + —-≡ - 5 =-+5—-7 ... f,

*n* I *n* - 1 τι +1 *2n* -1 *2n* +1 J

• τri, 1 1 1.1 I

π=τιsιn-1 14 rτ-n ï + o—ΓT∙∙∙ i^5

*n* I 7i - 1 ?i +1 *2n* - 1 *2n* +1 J

if we put *n* = 3, these become .=s√5(ι-∣+!-μi-∣+...).

3√3∕1 , 1 1 1 1 1 1 1 ∖

7r~ 2 ∖1 + 2 4 5 + 7 + 8'"∕

By differentiating (27) we get

, 1 1 1.1.1

cosec «Z/ — t, 4“. ∖o T . » « 4^, . x.> 4^√ ∩ ∖q\*,\*\*,

X2 (x + τr)2 *{X-* 7Γ)2 (x + 27γ)^ (ic-27r)2

putx=g, and we get τr2=9 {1 + 5∖ + y2 + jp +... }.

These series, among others, were given by Glaisher *{Quart. Journ. Math.,* vol. xii.).

We have sinh *ιrx = ιrxP(* 1 + æ■„ ), cosh *ιrx=P(* 1 + — —- ∣ ; if *∖ n2∕* ∖ 2τι + l∣2∕

we differentiate these formulæ after taking logarithms, we obtain the series

— cothτrz-^ = ii-j-^ + ^-i + 3-5-j-^+ ..., ⅛anh τrz= + gT-2 + gA-s+ ...

These series were given by Kummer (in *Crelle,s Joum.,* vol. xvii.).

111

The sum of the more general series -s s- + -s 1- -κ ≡-

° l2n + ie2n 22n + z2n 32" + x2"

+ ..., has been found by Glaisher *{Proc. Bond. Math. Soc.,* vol. viL).

If in the series (12) and (13) we put *n=2x,* Θ=^, we get

7rz-1 æ2 ι α⅛2(z2 -12) z2(z2-l2)(z2-22) cos-g--l -∣g+ ∣\* — + ...

sinψ=

These series were given by Schellbach (in *Crclle's Journ.,* vol. xlviii. ).

7Γ

If in the same series (12), (13) we put *θ=iς, n= —,* we get *A* 7Γ

4z2 4z2(4z2 - 2⅛2) 4z2(4z2 — 227γ2)(4z2 - 427γ2)

cos *x-l-* £-^2 + 1.2.3.4iγ4 i.2.3.4.5.6τr6 + ' ‘ ’

*2x* 2z(4z2 - 7r2) 2z(4z2 - 7γ2)(4z2 - 327γ2)

sιπæ- — - 1 2 3ιr3 + 1.2.3.4.5 τr5

We have of course assumed the legitimacy of the substitutions made. These last series have been discussed by Μ. David *{Bulk Soc. Math, de France,* vol. xi.) and Glaisher *{Mess, of Math.,* vol. vii.).

111

If *U,n* denotes the sum of the series p⅛+2^+g^+ · · · » *Pm* that of the series pn + ^+f^+ · · ·, and *JVm* that of the series 1~ - ~ *4-~^-2^+* ∙.∙, we obtain by taking logarithms in the formulæ (25) and (26)

log(xcos∞x)=C,(i) +⅛¾(≡) +g¾(⅛) + ...,

log (see ≈>= r,(⅛y+5r,(⅞),+lr.(⅞)∙+... i

and differentiating these series we get

2 t 2æ 1r2 τr4 ττ6

1 *V V. V6*

itanz=⅛f2⅛+ ⅛24z3+-2<⅛5 + (32).

2 7Γ2 7Γ4 7Γβ

In (31) *x* must lie between and in (32) between ±⅜τr. Write equation (30) iu the form

sec *x=∑{* -1)"7 ⅛⅛r--,

(2n + ⅛) -a≈2

and expand each term of this series in powers of a⅛2, then we get

*22JV1 2ilV3x2 2βJVsxi'ι ....*

sec *x=* + -7r3- + + (33),

where*x* must lie between ±⅜ττ. By comparing the series (31), (32), (33) with the expansions of cot *x,* tan *x,* sec *x* obtained otherwise, we can calculate the values of *U2, U4... V2, V4...* and *lV1, JV3....* When *Un* has been found, *Vn* may be obtained from the formula 2?K„=(2"-l)Un.

For Lord Brounker’s series of π, see Squaring the Circle (vol. xxii. p. 435). It can be got at once by putting α=l, 5 = 3, r · T≈, 1 > XV 111 1 α2 52

c = 5 . . .. in Euler s theorem = — τ-j .... = , =—....

*a b c a+ b-a+ c-b +*

Sylvester gave *{Phil. Mag.,* 1869) the continued fraction 5=1+J-L2 2l3 3λ4

2 1+ 1+ 1+ 1+ ”·’

which is equivalent to Wallis’s foraiula for 7r. This fraction was originally given by Euler *{Comm. Acad. Petropol.,* vol. xi.); it is also given by Stern (in *Crelle's Joum.,* vol. x.).

It may be shown by means of a transformation of the series for cos *x* and that tan *x= —* ... This may be also

*x* l-3-5-7- j

easily shown as follows. Let y=cos *∖jx,* and let *y,, y"..*. denote the differential coefficients of *y* with regard to *x,* then by forming these we can show that *ixy"* + 2y, + y=0, and thence by Leibnitz’s theorem we have

*4xfu +2) + {4n* + 2)y0i+1) 4. ^(ft) — θ.

Therefore — - 2 — - 2(2τι + l) ⅛x .

7/'“ *f∕y",* 7>+i)~ 2(2n + D *y(n+i)∕y(n+2y*

hence -2Vxcot Vaj=-2 —...

— 0 - — 10— — 14 —

Replacing Vx by *x* we have tan *x=^*— 5— —— · · ·

Euler gave the continued fraction

*, \_n* tan *x {rii* - 1 ) tan¾ (τι2 - 4) tan¾ (τι2 - 9) tanξ⅛ tanτ∞-- — — —— — —...;

this was published in *Mem. de l'Acad. de St Pétersb.,* vol. vi. Glaisher has remarked *{Mess, of Math.,* vol. iv.) that this may be derived by forming the differential equation

(1 - ar≈)√w+2> - *{2m* + l)√m+1> + (τι2 - τn2)√,n)=0, where y=cos *{n* arc cos *f),* then replacing *x* by cos *x,* and proceeding as in the former case. If we put 7i=0, this becomes

tan *x* tan⅛ 4 tan¾ 9 tan¾

X~ ΓΓ 3+-~5+ 7+~

whence we have

*x x2 4x2 9x2 n2x2*

arc tan æ=, —- s— ι— =—... +7; ≡—...

l+3+5+7+ 2τι + l +

It is possible to make the investigation of the properties of the simple circular functions rest on a purely analytical basis. The sine