The fundamental and overtones of the second either coincide with or fall midway between overtones in the first, and there is no approach to a dissonant frequency of beats, and the concord is perfect.

But obviously in either the octave or the fifth, if the tuning is imperfect, beats occur all along the line wherever the tones should coincide with perfect tuning. Thus it is easy to detect a want of tuning in these intervals.

The harshness of deep notes on instruments rich in overtones may be explained as arising from beats between successive over­tones. Thus, if a note of frequency 64 is sounded, and if all the successive overtones are present, the difference of frequency will be 64, and this is an unpleasant interval when we get to the middle of the scale, say to overtones 256 and 320 or to 512 and 576. Thus Helmholtz explains the jarring and braying which are sometimes heard in bass voices. These cases must serve to illustrate the theory. For a full discussion see his *Sensations of Tone,* ch. 10.

*Dissonance between Pure Tones.—*When two sources emit only pure tones we might expect that we should have no dissonance when, as in the major seventh, the beat frequency is greater than the range of harshness. But the interval is still dissonant, and this is to be explained by the fact that the two tones unite to give a third tone of the frequency of the beats, easily heard when the two primary tones are loud. This tone may be within dissonance range of one of the primaries. Thus, take the major seventh with frequencies 256 and 480. There will be a tone frequency 480—256 = 224, and this will be very dissonant with 256.

The tone of the frequency of the beats was discovered by Georg Andreas Sorge in 1740, and independently a few years later by Giuseppe Tartini, after whom it is named. It may easily be heard when a double whistle with notes of different pitch is blown strongly, or when two gongs are loudly sounded close to the hearer. It is heard, too, when two notes on the harmonium are loudly sounded. Formerly it was generally supposed that the Tartini tone was due to the beats themselves, that the mere variation in the amplitude was equivalent, as far as the ear is concerned, to a superposition on the two original tones of a smooth sine displacement of the same periodicity as that variation. This view has still some supporters, and among its recent advocates are Koenig and Hermann. But it is very difficult to suppose that the same sensation would be aroused by a truly periodic displacement represented by a smooth curve, and a displacement in which the period is only in the amplitude of the to-and-fro motion, and which is represented by a jagged curve. No explanation is given by the supposition; it is merely a statement which can hardly be accepted unless all other explanations fail.

*Combination Tones.—*Helmholtz has given a theory which certainly accounts for the production of a tone of the frequency of the beats and for other tones all grouped under the name of ‘‘ combination tones ” ; and in his *Sensations of Tone* (ch. 11) he examines the beats due to these combination tones and their effects in producing dissonance. The example we have given above of the major seventh must serve here. The reader is referred to the full discussion by Helmholtz. We shall conclude by a brief account of the ways in which combination tones may be produced. There appears to be no doubt that they are produced, and the only question is whether the theory accounts sufficiently for the intensity of the tones actually heard.

Combination tones may be produced in three ways: (1) In the neighbourhood of the source; (2) in the receiving mechanism of the ear; (3) in the medium conveying the waves.

I. We may illustrate the first method by taking a case dis­cussed by Helmholtz *(Sensations of Tone,* app. xvi.) where the two sources are reeds or pipes blown from the same wind-chest. Let us suppose that with constant excess of pressure, *p,* in the wind-chest, the amplitude produced is proportional to the pressure, so that the two tones issuing may be represented by *pa* sin 2ττn√ and *pb* sin *2πn2t.* Now as each source lets out the wind pcriodically it affects the pressure in the chest so that we cannot re­gard this as constant, but may take it as better represented by *p+λa* sin *(2πηit+e)+μb* sin *(2ιmit-{-f).* Then the issuing dis- turbance will be

(∕>-∣-λα sin *(2πnit+e)'Vμb* sin *(2τn2t+f)}{a* sin *2πnit+b* sin 2tγw2∕∣ *≈pa* sin *2πnJ+pb* sin *2πn2t*

. β2χ α2λ ∕ , 1 ∖

• + — cos *e——* cos (41rn√+0)

I *b2μ f b2μ l .,f∖*

+ — cos j—— cos (41rn2∕+∕)

+ cos (2π⅛-«2)/÷ß|— cos (2τr(wι-∣-Λ2)∕H-e}

+ 'ifι COS cos i2ιr(n∣+n2)z+∕∣ (35)

Thus, accompanying the two original pure tones there are (i) the octave of each; (2) a tone of frequency (*n*1 —*n*2); (3) a tone of frequency (*n*1+*n*2). The second is termed by Helmholtz the *difference tone,* and the third the *summation tone.* The amplitudes of these tones are proportional to the products of *a* and *b* multiplied by λ or *μ.* These combination tones will in turn react on the pressure and produce new combination tones with the original tones, or with each other, and such tones may be termed of the second, third, &c., order. It is evident that we may have tones of frequency

*hn1 kn2 hn1-kn2 hn1+kn2,*

where *h* and *k* are any integers. But inasmuch as the successive orders are proportional to λ λ2 λ3, or *µ μ2 μ3,* and λ and *µ* are small, they are of rapidly decreasing importance, and it is not certain that any beyond those in equation (35) correspond to our actual sensations. The combination tones thus produced in the source should have a physical existence in the air, and the amplitudes of those represented in (35) should be of the same order. The conditions assumed in this investigation are probably nearly realized in a harmonium and in a double siren of the form used by Helmholtz, and in these cases there can be no doubt that actual objective tones are produced, for they may be detected by the aid of resonators of the frequency of the tone sought for. If the tones had no existence outside the ear then resonators would not increase their loudness. There is not much difficulty in detecting the difference tone by a resonator if it is held, say, close to the reeds *of* a harmonium, and Helmholtz succeeded in detecting the summation tone by the aid of a resonator. Further, Rücker and Edser, using a siren as source, have succeeded in making a fork of the appropriate pitch respond to both difference and summation tones *(Phil. Mag.,* 1895, 39, p. 341. But there is no doubt that it is very difficult to detect the summation tone by the ear, and many workers have doubted the possibility, notwithstanding the evidence of such an observer as Helmholtz. Probably the fact noted by Mayer *(Phil. Mag.,* 1878, 2, p. 500, or Rayleigh, *Sound,* § 386) that sounds of considerable intensity when heard by themselves are liable to be completely obliterated by graver sounds of sufficient force goes far to explain this, for the summation tones are of course always accompanied by such graver sounds.

2. The second mode of production of combination tones, by the mechanism of the receiver, is discussed by Helmholtz *(Sensa­tions of Tone,* App. xii.) and Rayleigh *(Sound,* i. § 68). It depends on the restoring force due to the displacement of the receiver not being accurately proportional to the displacement. This want of proportionality will have a periodicity, that of the impinging waves, and so will produce vibrations just as does the variation of pressure in the case last investigated. We may see how this occurs by supposing that the restoring force of the receiving mechanism is represented by λx+∕*μ*x2, where *x* is the displacement and *μx2* is very small. Let an external force F act on the system, and for simplicity suppose its period is so great compared with that of the mechanism that we may take it as practically in equilibrium with the restoring force. Then. F = λx-j-μx2. Now μx2 is very small compared with λx, so that *x* is nearly equal to F∕λ, and as an approx­imation, F = λx-HzF2∕λ∖ or *χ* ≡≈ F∕λ — μF2∕λ3. Suppose now that F=α sin *2πnd-∖-b* sin *2πn2t,* the second term will evidently produce a series of combination tones of periodicities *2ni, 2η‰ nx — uì,* and nι+w2, as in the first method. There can be no doubt that the ear is an unsymmetrical vibrator, and that it makes combination tones, in some such way as is here indicated, out of two pure tones. Probably in most cases the combination tones which we hear are thus made, and possibly, too, the tones detected by Koenig, and by him named “ beat-tones.” He found that if two tones of frequencies *p* and *q* are sounded, and if *q* lies between N∕> and (N-{-ι)∕>, then a tone of frequency either (N-H)p — *o,* or of frequency *q—*N∕>, is heard. The difficulty in Helmholtzs theory is to account for the audibility of such beat tones when they are of a higher order than the first. Rucker and Edser quite failed to detect their external existence, so that apparently they are not produced in the source. If we are to assume that the tones received by the car are pure and free from partials, the loudness ot the beat- tones would appear to show that Helmholtz’s theory is not a complete account.

3. The third mode of production of combination tones, the production in the medium itself, follows from the varying velocity of different parts of the wave, as investigated at the beginning of this article. It is easily shown that after a time we shall have to superpose on the original displacement a displacement propor­tional to the square of the particle velocity, and this will introduce just the same set of combination tones. But probaly in practice there is not a sufficient interval between source and hearer lor these tones to grow into any importance, and they can at most be only a small addition to those formed in the source or the ear.

Bibliography.—For the history of experimental and theoretical acoustics see F. Rosenberger, *Geschichte der Physik* (1882-1890); J. C. Poggendorff, *Geschichte der Physik* (1879); and E. Gerland and F. Traurnüller, *Geschichte der physikalischen Experimentierkunst* (1899). The standard treatise on the mathematical theory is Lord Rayleigh’s *Theory of Sound* (2nd ed., 1894) ; this work also contains an account of experimental verifications. The same author’s *Scientific Papers* contains many experimental and mathematical contributions to the science. H. von Helmholtz treats the theoretical aspects of sound in his *Vorlesungen über die mathematischen*